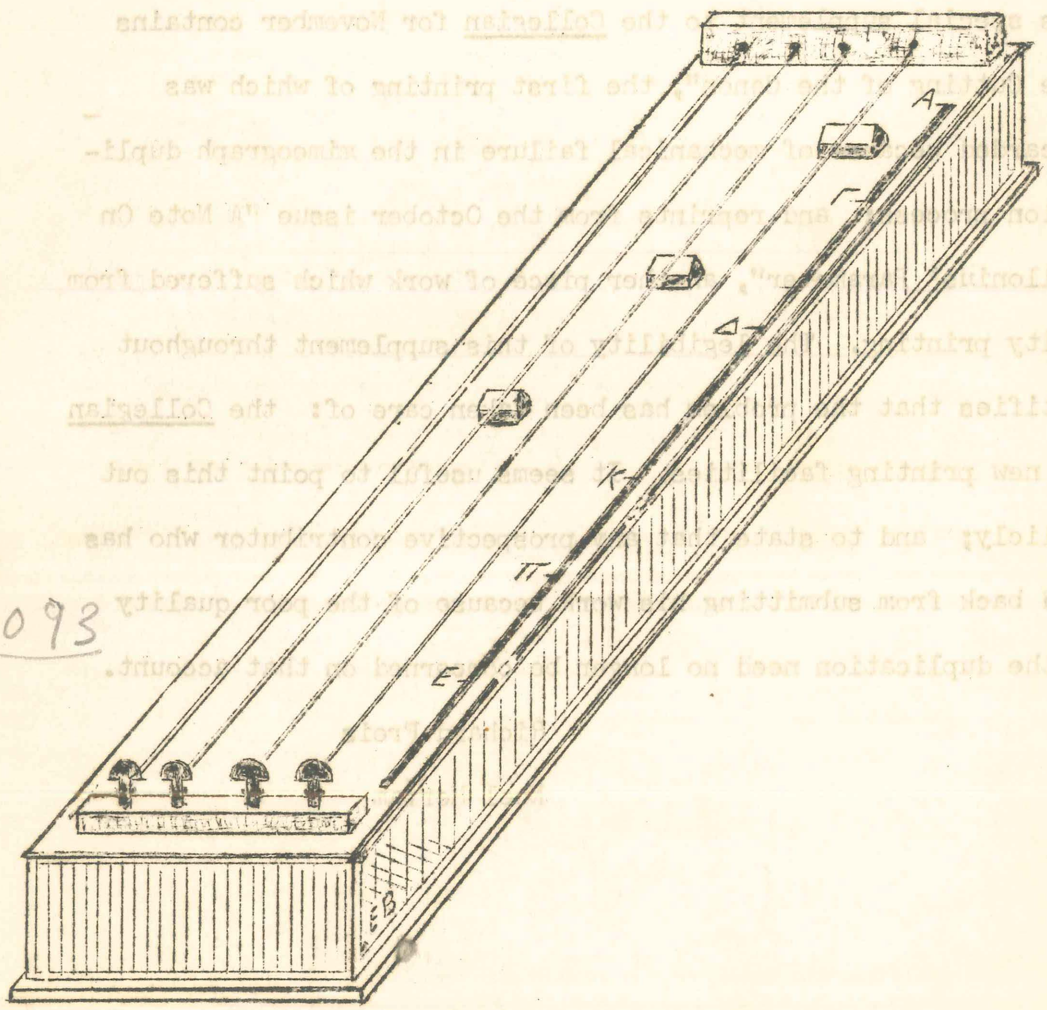

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St. John's College
The Collegian: Supplement
November, 1962

H. Foster
Leland
Mr. Elenor
Reigeldorfer.

This special supplement to the Collegian for November contains "The Cutting of the Canon", the first printing of which was discarded because of mechanical failure in the mimeograph duplication process; and reprints from the October issue "A Note On Apollonius' Parameter", another piece of work which suffered from faulty printing. The legibility of this supplement throughout testifies that the problem has been taken care of: the Collegian has new printing facilities. It seems useful to point this out publicly; and to state that any prospective contributor who has held back from submitting his work because of the poor quality of the duplication need no longer be concerned on that account.

Richard Freis

Noel Meriam

THE CUTTING OF THE CANON

Eva Brann

I would like to thank Mr. Freis for the great pains he has taken with my contribution to the <u>Collegian</u>.

E.B.

ΚΑΝΟΝΟΣ ΚΑΤΑΤΟΜΗ

(Sectio Canonis)

In the exercise on Musical Intervals in Chapter V of the Freshman Physics Manual the beginning of a little ancient work giving the mathematical basis for treating intervals in terms of string-lengths is quoted. Mr. Ossorgin suggested that it might be well to have the whole available in English:

A. THE CUTTING OF THE CANON

If there were rest (ἡσυχία) and motionlessness (ἀκινησία), there would be silence (σιωπή); if there were silence and nothing which was moved, nothing would be heard; for if anything is to be heard there must first be a blow (πληγή) and a motion (κίνησις). So that, since all sounds occur when a blow is struck, and it is impossible for a blow to be struck unless there is first motion - but of motions some are more frequent (πυκνότεραι) while others are more intermittent (ἀραιότεραι), and the more frequent motions make higher (ὀξυτέρους) sounds (φθόγγους) while the more intermittent ones make lower (βαρυτέρους) sounds - some sounds must of necessity be higher, since they are composed of more frequent and more numerous motions, while others must be lower since they are composed of more intermittent and fewer motions. So that the sounds which are higher than they should be are adjusted when they are relaxed (ἀνιεμένους) by a diminution of motion, and the sounds that are lower are adjusted when they are intensified (ἐπιτεινομένους) by an addition of motion. On account of this it may be said that sounds are composed (συγκεῖσθαι) of parts, since they are adjusted by addition and diminution. But all things which are composed of parts are said to be to each other in the ratio (ἐν λόγῳ) of a number (ἀριθμοῦ) to a number, so that sounds must also of necessity be to each other in the ratio of a number to a number. And of numbers some are said to be to each other in a multiple, some in a superparticular, and some in a superpartient ratio, so that sounds must also of necessity be said to be in such ratios to one another. Of these the multiple and the superparticular have a single name for their relation toward each other.

We know that of sounds some are consonant (συμφώνους) and others dissonant (διαφώνους), and that the consonant sounds make one mixture of both [sounds of an interval], while the dissonant sounds do not. Since this holds for them, it is likely that the consonant sounds, since they make one mixture of voice of both, should be expressible in ratios of those numbers which have a single name for their relation toward each other, whether they be multiple or superparticular.

* I.

<u>If a multiple interval, when doubly compounded, makes some interval, this interval will itself be multiple.</u>

Let there be an interval B-C, and let B be a multiple of C, and as C is to B, so let B be to D. I say that D is the duplicate of C. For since B is a multiple of C, C measures B. But as C is to B, so is B to D, so that C also measures D. Therefore D is a multiple of C.

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* The diagrams from the Manuscripts are on the <u>Diagram</u> sheet, at end of Section F.

II.

If an interval, when doubly compounded, makes the whole multiple, it will itself be multiple.

Let there be an interval B-C and as C is to B, so let B be to D, and let D be a multiple of C. I say that B is also a multiple of C. For since D is a multiple of C, C measures D. But we learned that, if there are however many numbers in proportion, and the first measures the last, it will also measure those inbetween. Therefore C measures B. Therefore B is a multiple of C.

III.

Neither one mean (μέσος) number nor more fall within the super-particular interval proportionate-wise (ἀνάλογον).

Let there be a superparticular interval B-C. And let the least numbers which stand in the same ratio B to C be DE, F. These then have only the unit as a common measure. Subtract BE equal to F. And since DE is a superparticular of F, the remainder DG is a common measure of both DE and F. Therefore DG is a unit. Therefore no mean will fall between DE and F. For the mean so falling will be less than DE and greater than F, so that the unit is divided, which is impossible. Therefore none will fall between DE and F. But as many means as fall between the least number proportionate-wise, so many will fall proportionate-wise between those [larger numbers] having the same ratio. And none will fall between DE and F, nor between B and C.

Errata to The Collegian

Supplement
November, 1962

p. 48: The reference of the gnomon to astronomy is inept; Philolaos probably means the arithmetical gnomon.

p. 63: Last paragraph - it should, of course be 2^6, cf. Nicomachus I,16.

p. g: Middle of page - ...by an area equal to the square on the abscissa reduced by the number equivalent to the ratio etc....

just below - ...the expression in the end will be

$$y^2 = P_e . x'' - \frac{P_e}{A''F} . x''^2 \quad .$$

There are indubitably many other errors.

IV.

<u>If a non-multiple interval is doubly compounded, the whole will be neither multiple nor superparticular.</u>

For let there be a non-multiple interval B-C, and as C is to B, so let B be to D. I say that D is neither a multiple nor a superparticular of C. For first let D be a multiple of C. Then we learned that, if an interval when doubly compounded, makes the whole multiple, it is itself also multiple (II). Therefore B will be a multiple of C. But it was not. Therefore it is impossible for D to be a multiple of C. But neither is it a superparticular. For no mean falls proportionate-wise within a superparticular interval (III). But B falls between D and C. Therefore it is impossible for D to be either a multiple or a superparticular of C.

V.

<u>If an interval, when doubly compounded, does not make the whole multiple, it will not itself be multiple.</u>

Let there be an interval B-C, and as C is to B, so let B be to D, and let D not be a multiple of C. I say that neither will B be a multiple of C. For if B is a multiple of C, then D will be a multiple of C. But it is not. Therefore B will not be a multiple of C.

VI.

<u>The double interval consists (συνέστηκεν) of the two greatest superparticular ratios, the sesquialter [3:2] and the sesquitertian [4:3].</u>

For let B-C be greater by a half than D-E, and D-E again greater by a third than F. I say that B-C is the double of F. For I took away

E-K equal to F and C-L equal to D-E. Then, since B-C is greater by half than D-E, B-L is the third part of B-C, and half of D-E. Again, since D-E is greater by half than F, D-K is a fourth of D-E and a third of F. Then since D-K is a fourth of D-Z, B-L is half of D-E, so that B-L will be half of D-K. And B-L was a third part of B-C. Therefore D-K is a sixth part of B-C. And D-K was a third part of F. Therefore B-C is the double of F.

Alternate

Let A be greater by a half than B, and B greater by a third than C. I say that A is the double of C. For since A is greater by a half than B, A contains B and its half. Therefore two of A are equal to three of B. Again, since B is greater by a third than C, B contains C and its third. Therefore three of B are equal to four of C. But three of B are equal to two of A. Therefore two of A are equal to four of C. Therefore A is equal to two of C. Therefore A is the double of C.

VII.

From the double interval and the sesquialter interval the triple interval (τριπλάσιον) comes to be.

For let there be A, the double of B, and B greater by a half than C. I say that A is the triple of C. For since A is the double of B, A is equal to two of B. Again, since B is greater by half than C, B contains C and a half of it. Therefore two of B are equal to three of C. Therefore A is equal to three of C. Therefore A is the triple of C.

VIII.

<u>If from the sesquialter interval the sesquitertian interval is taken away, the remaining interval which is left is the sesquioctavan.</u>

For let A be greater by a half than B, and C greater by a third than B; I say that A is the sesquioctavan of C.

For since A is greater by a half than B, A contains B and its half. Therefore eight of A are equal to twelve of B. Again since C is greater by a third than B, therefore C contains B and its third. Therefore nine of C are equal to twelve of B, and twelve of B are equal to eight of A. Therefore eight of A are equal to nine of C. Therefore A is equal to C and an eighth of it. Therefore A is the sesquioctavan of C.

IX.

<u>Six sesquioctavan intervals are greater than one double interval.</u>

For let there be one number, A. And let A be greater by an eighth than B, and B greater by an eighth than C, and C greater by an eighth than D, and E greater by an eighth than D, and F greater by an eighth than E, and G greater by an eighth than F. I say that G is greater than the double of A.

Since we learned to find seven numbers, each greater by an eighth than the preceding, let there be found A, B, C, D, E, F, G, and let A be 262144.

B	-	294912
C	-	331776
D	-	373248
E	-	419904
F	-	472392
G	-	531441, and G is greater than

the double of A.

X.

The diapason interval is multiple.

Let there be the nete hyperbolaeon A, the mese B, and the proslambanomenos C. Therefore the interval A-C, since it is double the diapason, is consonant. And so it is either superparticular or multiple. But it is not superparticular. For within a superparticular no mean falls proportionate-wise (III). It is therefore, multiple. Since then two equal intervals, A-B and B-C, when compounded make the whole multiple (II), A-B is also multiple.

XI.

The diatessaron interval and the diapente interval are each superparticular.

For let there be the nete synemmenon A, the mese B, the hypate meson C. Therefore the interval A-C, since it is double the diatessaron, is dissonant. Therefore it is not multiple. Since then two equal intervals A-B, B-C, when compounded, do not make the whole multiple (V), A-B is not multiple. And it is consonant. Therefore it is superparticular. And so also for the diapente.

XII.

The diapason interval is double.

For we showed that it was multiple (X). And thus, it is either double or greater than double. But since we showed the double interval to consist of the two greatest superparticulars (VI), so that, if the diapason is to be greater than double it will not consist of only two superparticulars but of more - and it consists of two consonant intervals, the diapente and the diatessaron - therefore the diapason will not be greater than double. Therefore it is double.

But since the diapason is double, and the double consists of the two greatest superparticulars (VI), the diapason consists of the sesquialter and the sesquitertian. For these are the largest. But it consists of the diapente and the diatessaron, which are superparticulars (XI). Therefore the diapente, since it is larger, is the sesquialter, and the diatessaron the sesquitertian.

Now it is clear that the diapente and the diapason are triple. For we showed that from the double interval and the sesquialter comes the triple interval (VII), so that the diapason and the diapente are also triple.

And the double diapason is quadruple (τετραπλασίον).

Therefore it has been shown for each of the consonant intervals what ratios the containing (περιεχοντας) sounds have to one another.

XIII.

It remains to be said of the tonic (τονιαῖος) interval that it is sesquioctavan [9:8].

For we learned that, if from a sesquialter interval a sesquitertian interval is taken away, the remainder which is left is sesquioctavan (VIII). And if from the diapente the diatessaron is taken away, the remainder is a tonic interval. Therefore the tonic interval is sesquioctavan.

XIV.

The diapason is less than six tones.

For it has been shown that the diapason is double (XII), while the tone is sesquioctovan (XIII). And six sesquioctavan intervals are greater than the double interval (IX). Therefore the diapason is less than six tones.

XV.

The diatessaron is less than two and a half tones and the diapente is less than three and a half tones.

For let there be the nete diezeugmenon B, the paramese C, the mese D, the hypaton mese E. Then the interval C-D is a tone, and B-E, which is a diapason, is less than six tones (XIV). Therefore the remaining intervals, B-C and D-E, which are equal, are less than five tones. So that there are less than two tones and a half in B-C, which is a diatessaron, and B-D is less than three tones and a half, which is a diapente.

XVI.

The tone is not to be divided into two equal parts or more.

For it was shown to be superparticular. And within a superparticular interval neither one nor more means fall proportionate-wise (III). Therefore the tone is not to be divided into equal parts.

XVII.

The paranetae and lichani are to be obtained through consonance (ληφθήσονται διὰ συμφωνίας) thus:

Let there be the mese B. Let the pitch be raised (ἐπιτετάσθω) a diatessaron to C, and from C let it be lowered (ἀνείσθω) a diapente to D. Therefore B-D is a tone. Again, from D let it be raised a diatessaron to E and from E let it be lowered a diapente to F.

Therefore F-D is a tone.

Therefore F-B is a double tone (δίτονος). Therefore F is a lichanos. Similarly also the parametae are to be obtained.

XVIII.

The parahypatae and the tritae do not divide the pycnum into equal parts. For let there be the mese B, the lichanos C and the hypate D. Let the pitch be lowered a diapente from B to F. Therefore F-D is a tone. And from F let it be raised a diatessaron to E. Therefore F-D, and also B-E, is a tone interval. And let D-C be added in common. Therefore F-E is equal to D-B. But F-E is a diatessaron. Therefore no mean falls proportionate-wise within F-E. For the interval is superparticular. And D-B is equal to F-E. Therefore no mean falls within D-C, which is the interval from a hypate to a lichanos. Therefore the parhypate does not divide the pycnum into equal parts. And similarly, neither does the trite.

XIX.

To mark off (καταγράψαι) the canon according to the so-called invariable scale (ἀμετάβολον σύστημα).

Let there be the length of the canon, which is also that of the string (χορδή) A-B, and let it be divided into four equal parts at C, D, E. Therefore B-A, which is the lowest sound, will be the 'boomer' (βόμβυξ, lowest flute tone). But this interval A-B is greater by a third than C-B, so that C-B will be consonant with A-B as a diatessaron upwards (in pitch - ἐπὶ τὴν ὀξύτητα). And A-B is a proslambanomenos. Therefore C-B will be a hypaton diatonos (διάτονος). Again, since A-B is twice B-D, it will be consonant as a diapason, and B-D will be a mese. Again, since A-B is four times E-B, E-B will be a nete hyperbolaeon. I cut C-B in two at F. And C-B will be double F-B, so that C-B is consonant with F-B as a diapason. So that F-B is a nete synemmenon. I took away the third part D-G from D-B. And D-B will be greater by a half than G-B, so that D-B will be consonant with G-B as a diapente. Therefore G-B will be a nete diezeugmenon. I set out G-H equal to G-B, so that H-B will be consonant with G-B as a diapason, and H-B is a hypate meson. I took H-K, the third part of H-B, and G-B will be greater by a half than K-H, so that K-B is a paramese. I took L-K equal to K-B and L-B will be a low hypate. Therefore all the stable (ἑστῶτες) sounds of the invariable scale will have been obtained in the canon.

XX.

<u>It remains to obtain the shifting</u> (φερομένους) <u>sounds.</u> I cut E-B into eight parts and I set out E-M equal to one of them, so that M-B is greater by an eighth than E-B. And again dividing M-B into eight parts, I set out one of them equal to N-M. Therefore N-B will be lower by a tone than B-M, and M-B than B-E, so that N-B will be a trite hyperbolaeon, and M-B a hyperbolaeon diatonos. I took the third part of N-B and I set out N-Ξ, so that Ξ-B is greater by a half than N-B and is consonant as a diatessaron downward (ἐπὶ τὴν βαρύτητα) and Ξ-B is a trite diezeugmenon. Again taking half of Ξ-B, I set out Ξ-O so that O-B is consonant with Ξ-B as a diapente.

Therefore O-B will be a parhypate meson. And I set out O-Π equal to I-O, so that Π-B is a parhypate hypaton. Now I took Γ [=C]-P the fourth part of B-Γ [=C], so that P-B is a meson diatonos.

B. Text

I have used: Euclidis Phaenomena et Scripta Musica, ed. H. Menge, Teubner texts. Facing Greek and Latin. Large parts of the treatise are quoted by Porphyry, Commentary on the Harmonics of Ptolemy, p. 98 f., and by Boethius, On Music, IV, ii.

C. Author

Euclid is said to have written an Elements of Music. In our manuscripts two short treatises are attributed to Euclid, the present Κανόνος κατατομή (Sectio Canonis), and an Εἰσαγωγὴ ἁρμονική (Introduction to Harmonics). The latter, is in a way, complementary to the former since it deals with the fitting together (ἁρμόζειν) of intervals into scales or systems (συστήματα), as the former deals with the establishment of these intervals (διαστήματα) themselves. Of these, the Harmonics is by one Cleonides; the author of the Canon, at least in its present form, it not certainly known (see Menge, p. XXXVII f. and P. Tannery, "Inauthenticité de la 'Division du Canon' attribuée a Euclide*" (Proclus de Lycie, Les Commentaires sur le premier livre des éléments d'Euclide, trans. P. van Eecke, p. 62, note 1.) I have not looked into the matter much, but since "ἐν τῇ τοῦ κανόνος Εὐκλείδου κατατομῇ" occurs as a reference in Porphyry 'Commentary on the Harmonics of Ptolemy (ed. Düring, p. 98, 19; 92, 30) I shall here speak as if he were the author, though it seems to me that the assumptions of the Canon are pre-Euclidian; see Section G.

D. Glossary (except Scale)

Canon - κανών ; ruler or measuring rod. The musical canon is one on which the musical intervals are mathematically marked off by notches, so that when it is laid alongside an undivided string, the string can be divided by means of a bridge so as to have the proper comparative pitch. Single-stringed instruments of this sort, called monochords, were apparently used to study pitch. The Κανόνος κατατομή justifies mathematically the division of the ruler itself. (See Prop. XIX, and Nicomachus II, xxvii, 1, ed. D'Ooge, note 1.)

Diapason - διὰ πασῶν ; "through all" the scale's tones; the interval from the first to the last note; the double interval; the octave.

* Memoires Scientifiques, III, p. 213 f.), but is now accepted to be Euclid.

Diapente - διὰ πέντε ; "through five" of the scale's tones, the interval of the fifth which exceeds the fourth by one tone.

Diatessaron - διὰ τεσσάρων ; "through four" of the scale's tones, the interval of the fourth. Four degrees up or down on the scale, which comprise in the diatonic scale, where the semitonal interval is a minimal, either five or six semitones, while in the enharmonic scale, where the minimal is smaller, the range in terms of semitones is wider. (See Scale under diatonic, enharmonic). These intervals are variously arranged within the diatessaron, depending on the genus and the part of the scale.

Diatone - διάτονος ; the whole-tone interval 9:8.

Doubly Compounded - δὶς συντεθέν , bis compositum; compounded to make a duplicate ratio. The "doubleness" or "twoness" of the ratio comes out directly in the algebraic notation for a:b cp. b:c, namely $(\frac{a}{b})^2 = (\frac{b}{c})^2$, see Thomas of Bradwardine, His Tractatus de Proportionibus, trans. H. Lamar Crosby, Madison, 1961, p. 20 f.

Double - διπλάσιος ; either the double ratio 2:1 or the duplicate ratio.

Duplicate - διπλάσιος, duplo maior; Euclid V, Def. 9: "When three magnitudes are proportional the first is said to have to the third the duplicate ratio of what it has to the second."

Greater by a half than - ἡμιόλιος ; "half (again) the whole", the number $\frac{3}{2}$; by a third than - ἐπίτριτος; "a third besides", the number $\frac{4}{3}$. The cumbersome English phrase, which is the Latin "duplo maior quam", has been used, indicating that the fact that ὁ Α τοῦ Β ἐστιν ἡμιόλιος does not mean $A = \frac{2}{3}B$, nor really $\frac{3}{2}A = B$, but describes A as containing the whole and a part besides of B, i.e. as a superparticular number.

Interval - διάστημα - "distance"; Theon of Smyrna, Exposition of Things Mathematical Useful for Reading Plato, ed. Hiller, p. 81, 6 f. defines: "Interval differs from ratio, since interval is that which is inbetween terms which are both of like kind and unequal, while ratio is simply the relation to each other of terms of like kind. And therefore there is no interval between equal terms, while their ratio to one another is one and the same, namely that of equality. The interval between unequals is one and the same from either [term] to the other, but the ratio is other and opposite from either [term] to the other."

Aristoxenus, Harmonics I, 15, l. 25 f. defines: "An interval is what is bounded by two sounds not having the same pitch. For interval seems, speaking generally, to be a certain difference of pitches and a place receptive to sounds higher than the lower of the pitches bounding the interval and lower than the higher of them. And the difference of pitches (τάσεις) is in being more or less [high-] pitched (τετάσθαι - "tensed")."

The most thorough discussion, with many pertinent quotations, is in Porphyry's Commentary on the Harmonics of Ptolemy (ed. Düring), p. 91 f. This passage is useful, and perhaps worth presenting a translation of, but since it is rather long and the Greek rather easy, I omit it here, but will be glad to help anyone interested in doing it.

Mixture of voice - κρᾶσις τῆς φωνῆς, temperatio vocis; in this context φωνή as distinguished from φθόγγος, sound, appears to mean composite, musical "voice" e. g. when an instrument "speaks" through several tones as distinguished from an isolated sound. Euclid, Introduction to Harmonics, p. 186, l. 7: "sound then, is the tuneful falling (πτῶσις ἐμμελής) of the voice on one pitch." Cf. Ptolemy, Harmonics, ed. Düring, p. 35, 26: "Consonance is the falling in the same way and the mixture of two sounds differing in height or depth."

Multiple - πολλαπλάσιος, multiplex; Euclid, V, Def. 2: "The greater [magnitude] is a multiple of the less when it is measured by the less"; VII, Def. 5: "The greater number etc.......". It is also the name of the ratio between such magnitudes or numbers. I have translated it either as noun or as adjective, whichever seemed smoothest.

Sesquialter - ἡμιόλιος ; the ratio 3:2, also $\frac{3}{2}$ as integer cf. superparticular.

Sesquioctavan - ἐπόγδοος ; the ratio 9:8, also $\frac{9}{8}$ as integer, cf. superparticular.

Sesquitertian - ἐπίτριτος ; the ratio 4:3, also $\frac{4}{3}$ as integer, cf. superparticular.

Superparticular - ἐπιμόριος ; "a part besides"; a number which contains in itself the whole of the number compared with it and some part (i.e. factor) besides. Also the name of the ratios which together with the multiple ratios form the most important group of ratios in music (see Ptolemy, Harmonics, I, 5, l. 13 f). The "greatest" superparticular ratios are 3:2 and 4:3. (See Nicomachus, II, ch. i ff.)

Superpartient - ἐπιμερής ; a number which contains in itself the whole of the number compared and more than one part (i.e factor) of it, e.g. $\frac{5}{3}$, $\frac{10}{6}$. Also these ratios; the superpartients form the second great class of ratios.

E. The Scale

Scale - σύστημα. Aristoxenus, Harmonics, I, 16 defines; "the scale is to be thought of (νοητέον) as something compounded (σύνθετον) of more than one interval (διάστημα)." These are the names of the tones (τόνοι , "tensions," pitches, or keys) and their meanings:

Proslambanomenos - προσλαμβανόμενος; tone "taken in addition."

Hypate - ὑπάτη (χορδή) ; the "highest" (string).

Parhypate - παρυπάτη ; "next to the highest."

Lichanos - λιχανός ; "forefinger," i.e. plucked with the forefinger.

Of these, the latter three belong both to the hypaton - ὑπάτων the highest, and again to the meson - μέσων , the middle tetrachord.

Mese - μέση (χορδή) ; "middle" (string).

Paramese - παραμέση ; next to the "middle."

Trite - τρίτη ; the "third."

Paranete - παρανήτη ; "next to the lowest."

Nete - νήτη (νεάτη) ; the "lowest."

Of these, the latter three belong both to the diezeugmenon-διεζευγμένων, "disjunct" (or in alternate system the synemmenon - συνημμένων , "conjunct") tetrachord and to the hyperbolaeon - ὑπερβολαίων, the "additional," lowest tetrachord. (See the Harmonics of Aristoxenus, ed. H.S. Macran, Oxford 1902, III 58, and Introduction, p. 10 f. for the difficult and somewhat useless subject.)

As the name of the lichanos shows, the tones are named according to their position on the stringed instrument and not according to pitch. The nete, for instance, although the highest in pitch, is the lowest in position. This makes good sense if we remember that pitch varies inversely with string length, so that the string lowest with respect to the number of its length is highest in pitch.

Genera - γενῆ (of scales): There are three kinds of scales; diatonic, chromatic and enharmonic (Aristoxenus, Harmonics I, 17 and 22 f.) For the tones named above have positions only relatively fixed, so that the intervals between them may be shifted. Here is the comparative distribution over two tetrachords or an octave, omitting the chromatic scale, which does not appear in the Canon:

Enharmonic

ditone $\frac{3}{4}$ tone $\frac{1}{4}$ tone diatone ditone $1\frac{1}{4}$ tone $\frac{3}{4}$ tone

nete paranete trite paramese mese lichanos parahypate hypate

diatone diatone diatone semitone diatone diatone semitone

Diatonic

The enharmonic scale is, therefore, constituted predominantly of multiples of quarter tones and ditones - δίτονοι, double tones, while the diatonic scale is predominantly constituted of the diatone - i. e. the interval 9:8, also called the διάστημα τόνος, "tone interval" (distinct from τόνος meaning pitch or key).

F. Synopses and Notes to Propositions

(ab : ab^2 designates an interval, e. g. B-C, where a, b, c are least numbers and non-multiples of each other. I have distinguished intervals from ratios by putting a hyphen rather than a colon, i. e. B-C, instead of B:C.)

I. The last term in the continuous proportion beginning with a:ab is a^2b. Therefore a measures it.

II. If the duplicate of an interval is a : a^2 b, the original interval was the multiple interval a : ab.

Boethius, IV, ii, gives a variant of this proposition interesting for the discussion of ratio presented in G, (references to Euclid added):
"If a multiple interval is duplicately multiplied (multiplicetur), the interval which results from that multiplication will be multiple. Let there be a multiple interval B-C and B, a multiple of what C is, and let C:B: :B:D (Euclid VI, 11; IX, 19). Since then B is a multiple of what C is, the term C measures what B is, be it twice, or thrice, or more (Euclid V, Def. 2). But C:B: :B:D. Therefore the term B measures what D is (Euclid VII, Def. 20); and for this reason also the term C measures what D is (Euclid IX, 11); therefore D is a multiple of what C is, and the interval D-C has resulted from the interval B-C being doubly compounded and coupled with itself and multiplied/duplicately (ex composito bis copulatoque sibimet et per binarium

multiplicato B-C intervallo). And the same can be proved in numbers. For let B, the double, be to C as the binary is to the unit, so that C:B: :B:D etc..... ".

Tannery, op. cit., p. 215, calls Prop. II a paralogism since it uses the unproved converse of what is implied in the preamble and used in Prop. I, namely that if an antecedent measures the first consequent it also measures all the others. Euclid VIII, 7 would justify the converse.

III. The root ratio of the superparticular $ab : a^2 b$ is the non-multiple interval a:b.

IV. Let a:b be an interval of primes, then so must its duplicate a:c be. Otherwise a = ab by II, which it does not, and also the mean falls within a:c which is impossible by III.

V. If the duplicate of a:b is a:c, b is a non-multiple of a, or the duplicate would be $a^2 b$.

VI. $2 = \frac{3}{2}$ cp. $\frac{4}{3}$.

VII. 2 cp. $\frac{3}{2} = 3$.

VIII. This is de-compounding or compounding with the inverse equivalent to division in numbers. $\frac{3}{2}$ decp. $\frac{4}{3} \equiv \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}$.

IX. $(\frac{9}{8})^6 = \frac{531\ 441}{262\ 144} > 2$, i.e. $\frac{9}{8}$ is to be 6 times compounded with itself.

X. Since the octave is from the highest (hypate) to the middle (mese) and again from the middle to the lowest (nete), compound these to get the double-diapason, consonant with the diapason. By the preamble it will therefore be multiple. Then by III it cannot be superparticular, and by II the compounding ratios are multiple.

XI. A-B and B-C are diatessaron intervals in the conjunct system. Their compound sounds dissonant, and is therefore non-multiple, so that A-B is non-multiple. But it is a consonance, hence superparticular.

XII. The diapason is double; for by X it is multiple, i.e. at least $\frac{2}{1}$, and by VI, $2 = \frac{3}{2}$ cp. $\frac{4}{3}$, which are the greatest superparticulars. If therefore the diapason does not consist of these it consists of more. But it consists of only two intervals, the diapente and the diatessaron. A porism adds that therefore the diapente = $\frac{3}{2}$, and the diatessaron = $\frac{4}{3}$.

XIII. The single tone interval is the diapente less the diatessaron. By VIII this is $\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$. This proposition brings out the simple mathematical relation of the Canon to intervals: subtraction or addition of sections of the canon is equivalent to division or multiplication of intervals.

XIV. diapason = 2, tone = $\frac{9}{8}$, so that $2 < (\frac{9}{8})^6$. In this and the following propositions particular tones which might contain the intervals under discussion are given by way of illustration. It should be kept in mind that the "tone interval" is one definite interval, i.e. 9:8, while the "tones" whose names are used are various pitches which contain the intervals.

XV. In the tone sequence B-C-D-E,

$$\text{B-E} = \text{diapason} = 2 < 6 \text{ tones} = (\tfrac{9}{8})^6$$

$$\text{less C-D} = \text{diatone} = \tfrac{9}{8} = 1 \text{ tone}$$

$$\text{equals } 2\ \text{B-C} < 5 \text{ tones}$$

so that B-C = diatessaron $< 2\frac{1}{2}$, and B-D = diapente $< 3\frac{1}{2}$.

XVI. Since no mean can fall between 9 and 8 without dividing the unit, $\frac{9}{8}$ cannot be decompounded.

XVII.

$$\overrightarrow{\text{diatessaron}}$$

$$\overleftarrow{\text{diapente}}$$

$$\text{F} \overset{\text{tone}}{\text{———}} \text{D} \overset{\text{tone}}{\text{———}} \text{B} \text{———} \text{E} \text{———} \text{C}$$

$$\overrightarrow{\text{diatessaron}}$$

$$\overleftarrow{\text{diapente}}$$

so that F-B is a ditone, which is the interval between nete and paranete in the upper tetrachord, and between mese and lichanos in the lower tetrachord of the enharmonic scale (cf. Scale). Tannery, op. cit., p. 214, observes that while Props. XVII and XVIII assume the enharmonic, Props. XIX and XX assume the diatonic scale. Since the enharmonic was the primary early scale (Aristoxenus, Harmonics I, 1), Tannery infers that XIX and XX belong to a later version of a pre-Euclidian Canon; their presuppositions coincide with those of the Timaeus, but are not, Tannery thinks, taken directly from the dialogue, but rather borrowed from Eratosthenes (third century B.C.).

"Obtaining through consonance"; a technical phrase (Aristoxenus, Harmonics II, 54 ff.) for the method of constructing the scale by succesively compounding the perfectly consonant intervals: diapason

(2-1), diapente (4-3), diatessaron (3-2), either upward or downward, but keeping within the interval of the double octave (i.e. 2:1 cp. 2:1 = 4:1). In this way the comparative string lengths even of dissonant (i.e. superpartient) intervals such as the diatone are found mathematically (precisely as is done in this proposition), and so the pitches of all the fifteen tones of the 'complete' (τέλεια) system. This method is presented in Ch. VI of the Freshman Physics Manual. The words λαμβάνω and λῆψις are used in Euclid (V, Def., 12 f.) for "taking" or constructing new from given ratios.

XVIII. $B \xrightarrow{\text{tone}} F \text{ --- } C \text{ --- } D \xrightarrow{\text{tone}} E$

diapente $\longrightarrow$

$\longleftarrow$ diatessaron

B-C is a diatessaron, and thus D-E = B-F, tones. And D-B = E-F, a diatessaron and thus a superparticular (XI) within which no mean can fall lichanos to hypate. Therefore the parhypate between these cannot form a sub-duplicate or half interval.

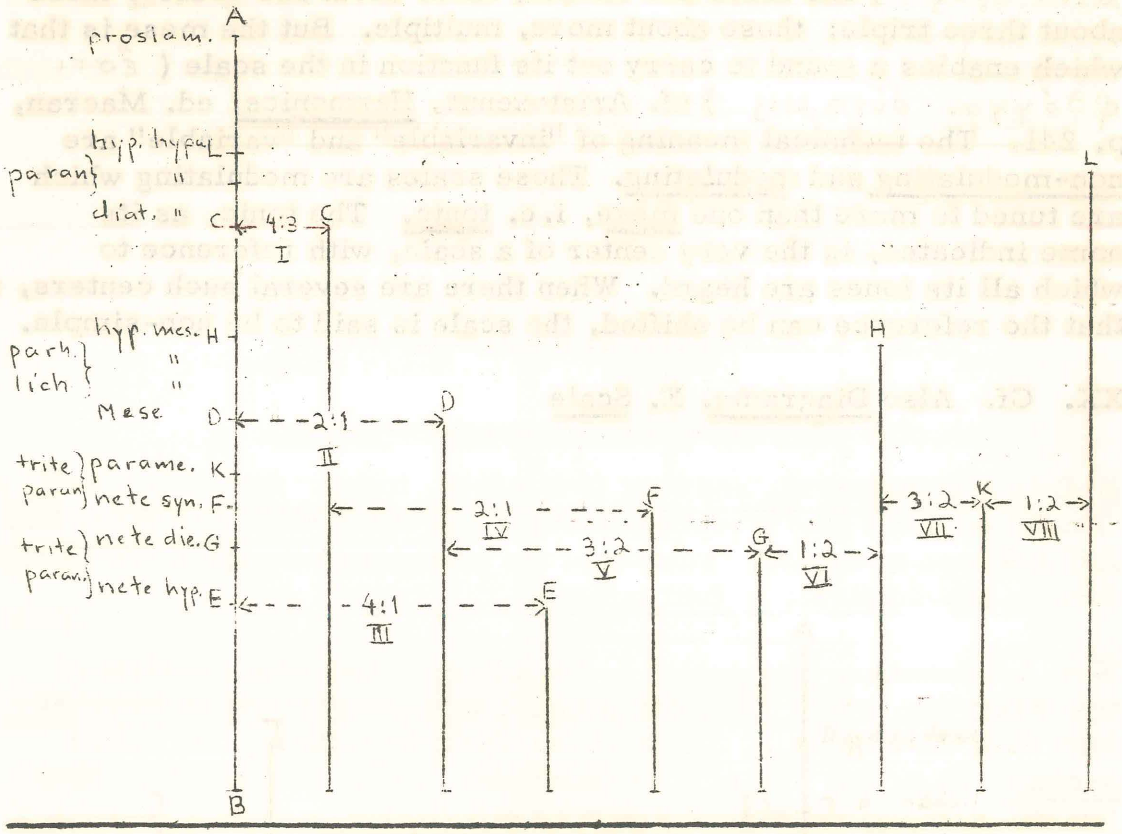
Now Aristoxenus, Harmonics I, 24 defines: "Let what consists of two intervals, which when compounded will comprehend an interval less than the remaining fourth [of the whole tetrachord], be called pycnum (i.e. the "dense", or uncuttable part of the canon)". Now for the upper enharmonic tetrachord (cf. Scale):

mese $\xrightarrow{\text{ditone}}$ lichanos $\xrightarrow{1\frac{1}{4}\text{ tone}}$ parhypate $\xrightarrow{\frac{3}{4}\text{tone}}$ hypate

$$\left(\frac{9}{8}\right)^2 > \frac{5}{4}\cdot\frac{9}{8} \quad \text{cp.} \quad \frac{3}{4}\cdot\frac{9}{8},$$

so that lichanos - hypate is a pycnum, within which the parhypate is not a mean.

XIX. Cf. Also Diagrams, E. Scale.



I. diatessaron:	proslambanomenos	- lichanos hypaton
II. diapason:	proslambanomenos	- mese hypaton
III. double diapason:	proslambanomenos	- nete hyperbolaeon
IV. diapason:	lichanos	- nete synemmenon
V. diapente:	mese	- nete diezeugmenon
VI. diapason:	nete diezeugmenon	- hypate meson
VII. diapente:	hypate meson	- paramese
VIII. diapason:	paramese	- hypate hypaton

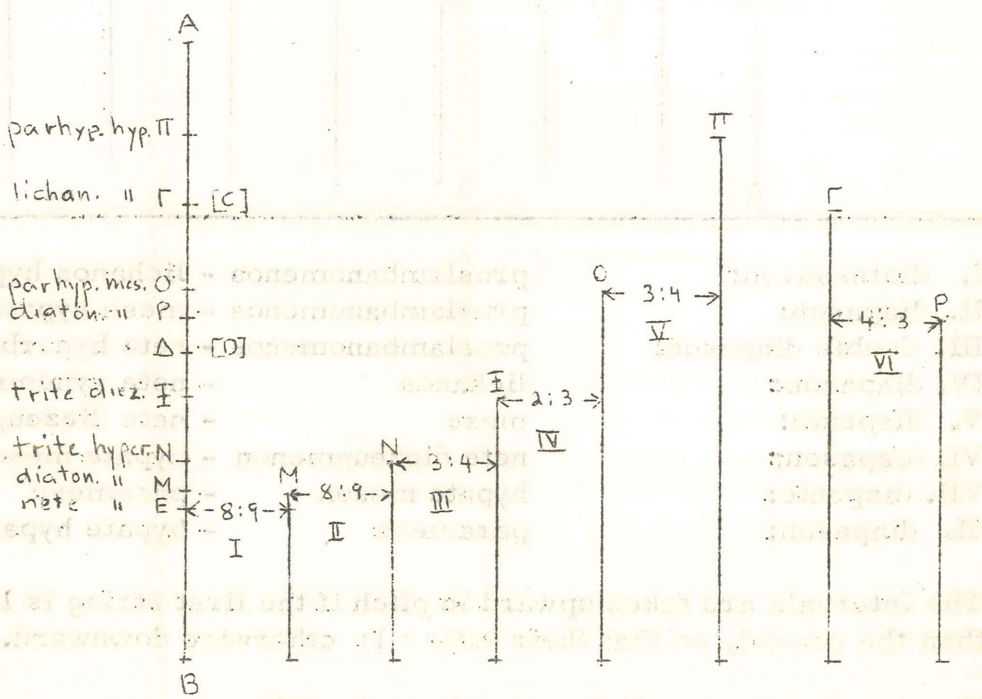
The intervals are taken upward in pitch if the first string is longer than the second, so that their ratio > 1; otherwise downward.

Hypaton diatonos = lichanos, cf. under XX.

The "stable" sounds: [Euclid], Introduction to Harmonics, 4, 1. 2 f. defines: "...stable (ἑστῶτες) are those sounds which do not change with the differences in genera, but remain at one pitch (τάσις). Movable (κινούμενοι), on the other hand, are those which are affected in the opposite way". He then gives a list corresponding to Props. XIX and XX.

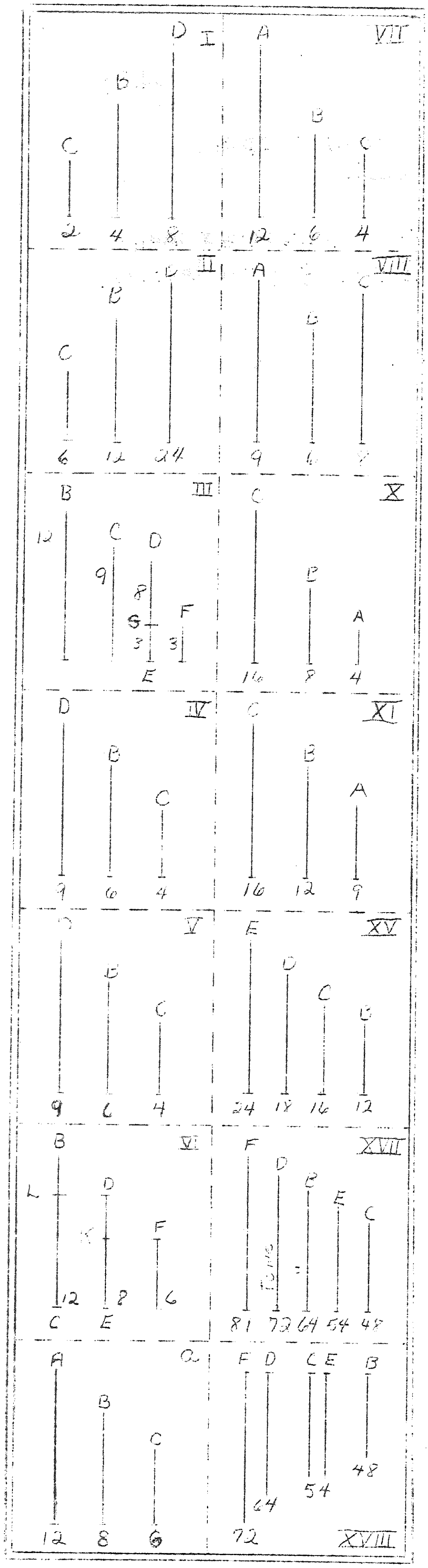
"Of the invariable scale": ibid, 11, 1. 1f. "The invariable (ἀμετάβολον) scale differs from the variable scale, as simple scales from those which are non-simple. Now those which are fitted together about (ἡρμοσμένα πρός) one mese are simple, those about two double, those about three triple; those about more, multiple. But the mese is that which enables a sound to carry out its function in the scale (ἔστι ..... φθόγγου δύναμις) cf. Aristoxenus, Harmonics, ed. Macran, p. 241. The technical meaning of "invariable" and "variable" are non-modulating and modulating. Those scales are modulating which are tuned to more than one mese, i.e. tonic. The tonic, as its name indicates, is the very center of a scale, with reference to which all its tones are heard. When there are several such centers, so that the reference can be shifted, the scale is said to be non-simple.

XX. Cf. Also Diagrams, E. Scale

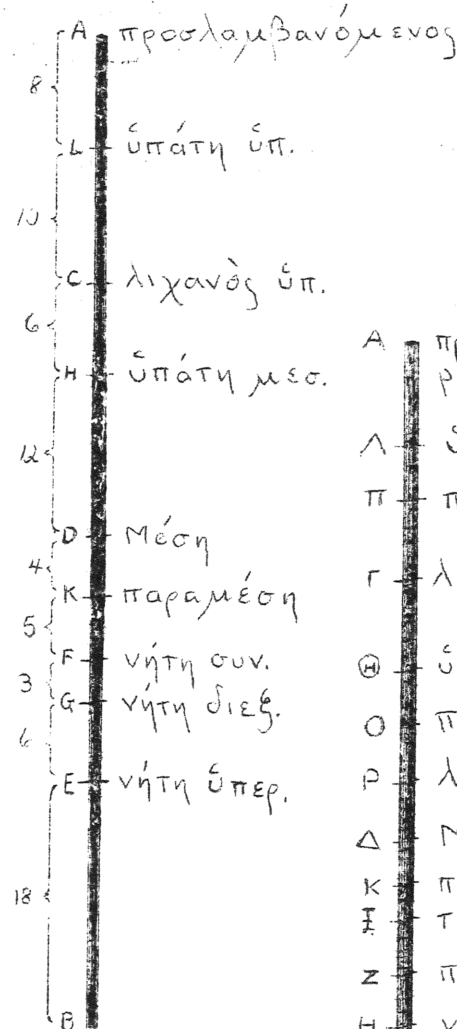


I.	tone:	nete hypaton	- diatonos hypaton
II.	tone:	diatonos hypaton	- trite hypaton
III.	diatessaron:	trite hypaton	- trite diezeugmenon
IV.	diapente:	trite diezeugmenon	- parhypate meson
V.	diatessaron:	parhypate meson	- parhypate hypaton
VI.	diatessaron:	lichanos hypaton	- diatonos meson

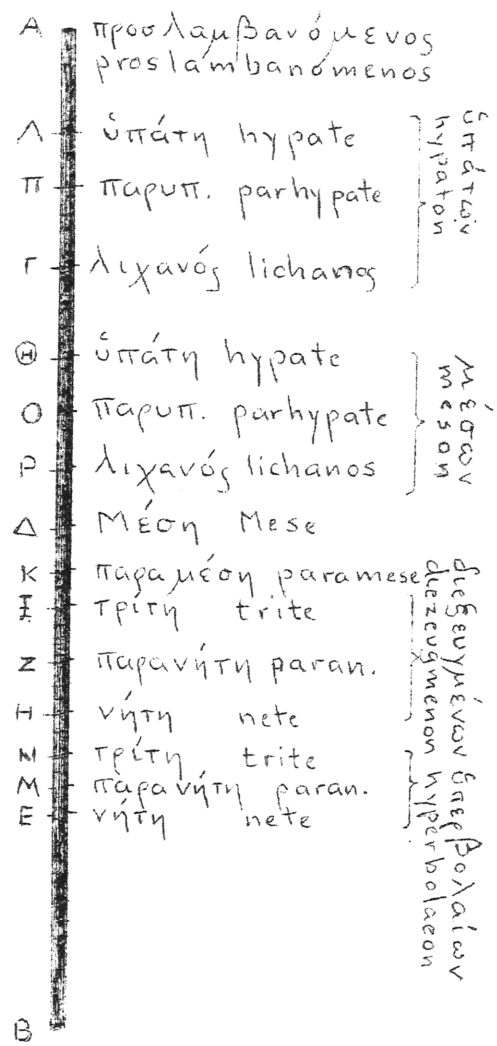
The Canon:



Propositions
I-XVIII



from Prop. XIX*



from the margin
of a manuscript

*The numerical divisions in terms of a line of 72 parts have been added.

ΚΑΝΟΝΟΣ ΚΑΤΑΤΟΜΗ

Diagrams

Again, since all the string ratios (except the last) < 1, the intervals (except the last) are downward in pitch.

Diatonos meson = lichanos; diatonos hyper. = paranete for the diatonic scale, see [Euclid], *Introduction to Harmonics*, 4, ls. 1, 4, 14.

For the "shifting" tones see "movable" tones, under XIX.

G. Interval and Ratio

[The texts on which the remarks which follow are based are quoted and/or translated in Notes to F. and G.]

Intervals may be thought of as sensible ratios.[1] What this means and how it bears on mathematics and the world -- and their mutual relation -- will be the matter of this section.

Let me begin with some observations about intervals and ratios as they occur in the Cutting of the Canon.

In the preamble to the Canon it is said that 1. since sounds are adjusted by addition and subtraction they are constituted of parts (ἐκ μορίων); 2. all things constituted of parts have to each other the ratio of a number to a number. Now in the light of the Eudoxian[2] general theory of proportion as set out in Book V of Euclid's Elements these statements are not completely acceptable, for 1. while it belongs to the very nature of magnitudes which have a ratio to one another to be adjustable by addition and subtraction of their own parts (Book V, Def. 4), it is also the very central matter of Book V that they need not be constituted of each other's parts -- but this is presumably what the preamble intends them to be[3]; 2. on the other hand, only commensurable magnitudes do bear to each other the ratio of a number to a number (X, 5) The writer of the Canon either ignored or was ignorant of all this. The Canon is thus written in the context of the world before the "scandal" of that offense to human speech, namely the "unspeakableness" of certain parts of the world in terms of others-- the fact of incommensurability -- broke into the open;[4] this means, broadly speaking, that the mathematics of the Canon is free to, and does, pre-suppose a certain direct and immediate view of the world, namely the belief that the world itself is learnable (μαθητόν), that it can be itself, literally, even bodily, caught in an account, and that this account is mathematics.

Now it seems to me that there are several places in mathematics where the more clearly things and what is done to them are defined and elaborated mathematically, the more obscure become the things and what is done to them as objects in the world. Perhaps this is true for all mathematics and happens at the very beginning of the science, but I mean nothing so radical here. Rather, having assented to and in some way understood the beginnings, I find that there are certain places where new departures are suddenly taken which are not quite explicable in terms of what preceded. I am thinking in particular of 1. the formation of proportions of ratios between different kinds of magnitudes; 2. the operation of compounding ratios. It seems to me that the notions which lie in and behind the Canon make these two matters more understandable than those which are found in the Elements. Clearly, the investigation of this will in the end also be useful for thinking about the relation of "technical" mathematics to the world. (By "world" I mean what I live in and by[5] -- just what is usually meant.)

It is necessary to set out what the writer of the Canon, had he thought about it, must have meant by ratio (λόγος - logos).

The first use of λέγω, the verb from which logos comes, which is cited in the Greek lexicon (Liddelland Scott) is Iliad, XXIII, 239[6]: "then let us gather (λέγωμεν) the bones of Patroclus the son of Menoetius"---from the ashes of the pyre, Achilles means. In Greek (and in English too), "picking out and gathering up" thus have the same name as speaking and reasoning, for this is what commonly means. For sorting out like things and collecting them are human deeds within the world which are very naturally, and perhaps, of necessity, accompanied by counting up, naming "the number of", telling "the tale", giving an "account"; and these latter phrases show how this kind of speech is taken as fundamental to any speaking, be it explaining, or excusing by giving reasons, or even story-telling.[7] This connection, in deed and in word, between the number of things (ἀριθμός -arithmos) and speech (λόγος) lies behind (though perhaps not in) the "purely" mathematical notion of ratio.

Now what it is in the being of things that the language takes note of in these coincidences of name, has several --- not very many--- intelligible formulations. I shall briefly describe that --naive[8] -- one which seems to me to underlie the Canon.

To someone who wonders what he is doing when he counts things, the thought might first come that he is doing nothing but repeating in and accompanying with his thoughts the doings of his hands.[9] But if he asks himself why his hands should be able to grasp distinct things and gather them into collections, he might come to think that this must be because these things had been distinguished and collected for him beforehand, and that his hands were, in fact, *repeating* the doings of thoughts. He might conclude that things in the world could not be really different from thoughts, and that he could account for his ability to handle and learn about the world best by supposing that whatever he grasped, be it by hand or in thought, must be a thought thing (νοητόν). But since he began by trying to understand his ability to count, he would suppose these thoughts --- and therefore thinkable --- things to be like that which is both made by and used in counting, namely ἀριθμοί --- numbers. Then, having noticed that story-tellers love long counts and seem to consider these essential to their tale (as for instance Homer in the *Catalogue of Ships*, (*Iliad II*, 484 f.) and that kings require them (*Republic* 522 d) --- having noticed also those coincidences of words for counting and speaking, and seeing many other such pointers, he might ask himself to what they point. Then it might come to him that the very words which have caught his attention (λόγοι), are a distinguishing and grasping hand to his thoughts, and that his thoughtful, reasoned account (λόγος) is the gathering and col-*lecting* embrace. And finally it might come to him that only in being spoken by him and heard by another do his logoi seem to become truly themselves. And so he would have the rudiments of an account of what there is, an account which closely connects word, number, account and speech.---The order of thinking, the words, and the tenor of feeling in this sketchy description are debatable, but <u>some</u> such cluster of insights must be kept in mind for what follows.

To return to logos in mathematics: When two people feel friendship toward one another, it will not be too much to say that two conglomerations of intractably <u>unlike</u> parts are joined in a relation which has inexhaustibly many respects --- but when two numbers have a ratio to one another, two magnitudes of <u>like kind</u> (ὁμογενῆ) are in a sort of relation in respect to size (κατὰ πηλικότητα ποιὰ σχέσις, Euclid, V, Def. 3). A relation between numbers is therefore very different from a human relation, or so it would seem. For friendship, or what is between friends, is distinguishable from the friends themselves, and those who are friends are not, usually, "by themselves" what they are toward their friends".[10] But what is between numbers, that is, their ratio, is neither more nor less than some sort of "hold" or bearing (σχέσις, <u>from</u> " ἔχω " <u>to</u> hold" - the German words: <u>Haltung</u> "bearing" --- <u>Verhältniss</u> "ratio" are just right) which holds them in a certain order of pre-cedence and which "relates" them in the double sense of "collecting" and "telling" them. The ratio relation of 2 to 1 is exhaustively stated when I put down 2:1. The terms held together in this way <u>are</u> the logos --- there is nothing in the logos but they themselves.[11] The logos is simply the "address" that numbers have toward each other: " number

makes all things have address toward each other (προσήγορα ἀλλήλοις",
Philolaos (Diels I, p. 411, 19). One of the terms is, to go even further, a part or parts _of_ the other, so perfectly does their relation absorb them. Furthermore, although, as is usually conceded, not all people can be friends with one another, every number can be in ratio with every other number, and only the non-numbers _nothing_ and _the infinite_ (neither of which _are_ anything) are excluded from this universal bond.[12] The kind of account, therefore, which is a λόγος ἐν ἀριθμοῖς, a logos in numbers, is peculiarly complete in itself and thoroughgoing in application. This suggests from the side of mathematics that it might be desirable to attempt an account of things in numbers, that things are best told in this way. The preamble of the _Canon_ was written by someone who is very much aware of the "telling" nature of the numerical logos --- four times he uses the somewhat redundant phrase ἐν λόγῳ λέγεσθαι-" _said_ to be in ratio...". Furthermore those ratios are most important for him which can be said with _one name_ (ἑνὶ ὀνόματι), which, as it happens, is the name of the antecedent number of the ratio (see Section D, _Superparticular_).

The _Canon_, then, states that sounds are to each other as these ratios of numbers. As it turns out, it is not only the reason given for this assumption which is faulty, but the assumption itself is not quite correct. Wonderfully enough it is the human speaking voice (λογικὴ φώνη) itself which, being continuous in pitch rather than discreet, would, if expressed in string-lengths, yield some incommensurable lengths[13] --- and so the human logos is, as it were, the very embodiment of the refutation of the power of the numerical logos. But this does not touch the immediately important assumption of the Canon, which is more special but also more remarkable, namely that sounds which go well with one another, so-called consonant sounds, have to each other the ratios made of the smallest numbers, and that all the other "musical" sounds, consonant or disconsonant, are in number ratios.

What happens between sounds is called "interval" (διάστημα) and is, in definitions, distinguished from the ratio relation (see Interval, Section D). An interval is, in fact, that which is contained inbetween the terms of the logos (Prop. XII: τοὺς περιέχοντας φθόγγους, "the containing sounds"). It is a distance across which one term faces another, gives the other its effectiveness and, quite literally, "effects" the other by arousing it as overtone.[14] It is something through which - one might even translate "because of which"-consonances are heard, be it the diapason (διὰ πασῶν ; "through all", the octave) or a smaller interval. The interval is something, something primarily heard, also seen, and lastly thought. The distinction between the relations of numbers and those of sounds is made by the ancients in a variety of ways. The Pythagoreans do not make it.[15]

What intervals are, in fact, consonant, and what their ratios are is again an assumption of the Canon. The consonances are: diapason (octave, 2-1), diapente (fifth, 3-2), diatessaron (fourth 4-3). It is the business of the Canon to show that these consonant intervals can be "put together" or compounded (συντέθῃ) in such a way as to "take" (λαμβάνειν) all the "aisthetically" possible sub-intervals within the diapason. When all the interstices that are there have been filled by this "joining" (ἁρμόζειν) process, the result is a close-fitted (ἡρμοσμένον - Introduction to Harmonics, 1, 2) whole, a system (σύστημα), as the Greeks called the scale. What interstices are there depends, again, on the ear.

But what is this compounding by which the scale is joined, and what is it that is compounded? It is notable that no attempt is made in the Canon to explain this. It is simply done. An appeal to the Elements on this matter would not, in any case have done any good, supposing it had been possible time-wise. Euclid gives neither a definition of what the operation is, nor particular instructions for how to do it. To be sure, particular compound ratios, namely those compounded of the same ratios, are named in Book V Def. 9-11; e. g. if A:B::B:C, then A:C is called the duplicate of A:B, while Prop. 20 shows that the duplicates of same ratios are again in proportion. Also, Book VI, 23 shows that there are things which have to each other ratios compounded of other ratios, namely equiangular parallelograms; these have to each other ratios compounded of the ratios of their sides, so that compounding is shown to be a lever from dimension to dimension, a source of growth.[16] By implication this might be said to show one how to compound --- but beyond these mere facts not much is made explicit. What in the world can it mean to "put together" ratios, to compound, to "double" or "triple" a mathematical relation? What is "relating relations"? (Among non-mathematical objects this is not always so dark a doing. For instance, for human beings "taking the duplicate" is a perfectly vivid occurrence. The relation " father: son" is duplicated as the son becomes a father; the resulting duplicate relation is "grandfather : grandson".)

The operation of compounding is indeed the subject of a definition (Euclid VI, Def. 5), which is, however, with good reason thought to be spurious.[17] Clearly it was added only because a lack was felt. It runs: "A ratio is said to be compounded (συγκεῖσθαι) of ratios when the sizes of the ratio multiplied together (πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι) λασι make some ratio". But there is and can be nothing in Book V to permit multiplication of ratios, for they are not numbers and cannot be subject to number operations. The definition is a kind of council of despair.

In view of this, what is done in the Canon? The Canon compounds intervals, not ratios. The trick is striking, for the preamble speaks only of sounds in number ratios and the propositions deal only with intervals.[18]

The word used for compounding, συντιθέναι "to put together", is the same as that used for the addition of parts; that for decompounding, or compounding with the inverse, is ἀφαιρεῖν, "to subtract"; and generally the language of compounding is the same as that of adding and subtracting.[19] For ratios this is puzzling, but for intervals it makes good sense. Intervals are simultaneously added and compounded The diapente (fifth, 4-3) subtracted from the diatessaron (fourth, 3-2), the last bounding tone of the former being the first bounding tone of the latter, gives the tone interval (9-8) in its mere position on the scale. But if I wish to make the interval effective as a consonance or dissonance[20], to "fit" it securely into the system, i.e., to find the string-lengths which will produce it, I must compound the intervals. How?

I can compound these intervals by successively altering the partitioning of the containing string-lengths. I begin with a string divided into 4 parts and another of 3, and between these the diapente sounds. I take the latter, regarding it now as a string of 2 parts --- so that each of the new parts will be greater by a half than each of the old. Then I "take" the third string of 3 parts --- which are found to be greater by a half than the two of the middle string. String I and III sound the tone interval, a dissonance. In numbers:

	I	II	III
	4	$3 \rightarrow \frac{3}{2}$	$4\frac{1}{2}$
or	8	6	9

To avoid comparative numbers (see Section D, Superparticular) like the hemiolion of String II, I can, by Euclid VII, 34, find the least common multiple of 3 and 2, and write the proportion with simple numbers.

When these successive changes in the bounding ratios have been made, the intervals will have been compounded. In the Canon itself, not three strings but a monochord with a movable bridge is used: In Prop. XIX we are finally told how to "engrave" (καταγράψαι) the canon on the length (μῆκος) of board underlying a string (χορδή). Compounding on a monochord retains more clearly the nature of the operation, namely the successive re-partitioning of the bounding term (ὅρος)[21] --- which turns out to be ultimately precisely the ability of a number to participate first in one ratio and then in another.

The things compounded and the way in which they are compounded are, therefore, in summary: 1. heard consonances or dissonances --- compounded by the operation of addition; 2. lengths of strings --- compounded by the operation of successive repartitioning; 3. ratios of numbers --- compounded by some undefined mathematical operation. But this mathematical operation seems to be unavoidably under the direction of the ear, as the language of Prop. XIX confirms, where the canon, or "ruler" is "ruled" off according to the deliverance of the ear: "I cut C-B twice . . . so that C-E will be consonant with E-B"; or: "therefore all the sounds in the canon will have been taken". It is hard to see how it can be otherwise.

While there are certainly objects within (Euclidian and Eudoxian) mathematics which have to each other a ratio compounded of others, especially those objects which have grown out of others of a lesser dimension, this is not the way in which they are primarily thought of --- hence their own genesis and with it the coming to be of compound ratios remains something not quite explicable within the mathematical context. Within the world, on the other hand, certain items which are especially appealing to the senses, such as musical intervals, are found to be compounded, i.e., to be the results of compounding ratios; indeed in the next Section the world itself will appear as such a compound --- an incarnate compound. And here, within the world, compounding is a sensible operation in which the eye can see the process and the ear hear the result, a new interval. In fact, in the musical context this operation is called "cutting the canon" (κανονίζειν).[22]

Here we return to the views which the writer, albeit without being aware of it, incorporated in the Canon. He succeeds in making fairly understandable an operation obscure in the "pure" mathematics of the Elements. He can do this because he does not distinguish between sounds bounding an interval and numbers set in a ratio, taking the thing heard and the thing counted as one. Neither does he distinguish those relations between things which are more or other than those very things (like consonance between sounds) from those relations which are nothing but the confrontation of these things (like ratios of numbers), thereby showing that all things are to him as numbers (or the reverse: it is indifferent). In short, he writes like a Pythagorean.

To go on with the matter of proportions of different magnitudes: The temptation, when compounding, to write simply:

$\frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}$, that is, to multiply, is great, although with respect to ratios it is a meaningless thing to do. The Greeks did succumb to it, as the spurious Def. 5 of Book VI shows. It is encouraged by several facts: 1. compound ratios have the names of multiples (e.g. διπλάσιος : duplicate or double); 2. the multiple names suggest exponents, i.e., the duplicate = $(\frac{a}{b})^2$; 3. compounding has the same effect of raising the dimension as multiplying numbers has (Euclid, Book V, 23 and VII, 16); 4. most important, the very name of certain ratios is the same as that of their antecedent numbers, i.e., ἡμιόλιος means the "comparative" number $\frac{3}{2}$ and the ratio 3:2. Why not, then, take the logos as an arithmos and use it accordingly? This is, in fact, what happens in Boethius's rendition of Prop. II of the Canon (see Section F, Notes on Prop. II). He cannot settle in his language whether he means to multiply or to compound. But he clearly, more clearly than the Canon, distinguishes between comparing intervals and number ratios.

More explicitly:

In this latter distinction, not yet necessary in the Canon, is indicated the reason why logoi are not simply converted into arithmoi, why a millenium later Galileo still retains ratios, and devotes the latter pages of the sixth day of his Two New Sciences to finding a replacement for Euclid VI, Def. 5, which in his admiration for Euclid, he cannot believe is genuine.[23)] The reason is the wonderful fact --- and here this word "fact" is perhaps, for once, used rightly --- that the world is "in proportion" -- not in the narrower mathematical sense of having parts which are magnitudes in "equal"[24)] ratios, but in the stranger sense that is proves feasible to say that motions, sounds, strings and numbers have all the same ratio throughout, are- ἀνά-λογον, though as objects they seem in no way the same.

Or perhaps it would be better to say that same ratio, i.e., proportion, is the bond between mathematics and the word --- e.g., the bounding tones of intervals have the same ratio as certain numbers, while the bond that ties the world in with itself is compound ratio --- e.g., the tones of the scale or "system" are taken by compounding consonances. (This striking observation about the world --- that its parts, usually against all expectation, are not in simple proprotional relation, seems to be the beginning of all "science": the first of the Two New Sciences begins with a refutation of the traditional and obvious assumption -- a refutation which comes on Sagredo as a "lightening-flash" [Dover, p. 3] ---

that the sizes of bodies are in simple proportion with their strengths; the second science, again, begins by establishing the fundamental compound relation which describes natural downward motion, namely that in "free fall" distance is covered in the duplicate ratio of the time taken, or, expressed in the notation of a modern counterpart of ancient compounding, joint variation: $s \propto t^2$ [Dover, p. 174].)

Now even within the "pure mathematics" of Euclid's Book VI there arises a certain difficulty about what can be in proprotion, for while Def. 3 of Book V does not demand more than that the magnitudes of any one ratio should be of the same kind, Prop. 16 provides for the possibility of alternating any proportion. What then is to happen in Book VI, where ratios of magnitudes not of the same kind, such as lines and areas, occur in proportion. If such proportions are alternated non-homogenous magnitudes like lines and areas will appear in ratio, and this is simply "impossible" (Theon, p. 73, 18-19). So that there is a conflict between what is permitted and what is possible. This difficulty, as well, is, though a small inconsistency within the Elements, a great perplexity at the foundations of mathematical physics, for the fact of "mixed" proportions is, as we have just seen, the very condition which makes this science possible, though it itself remains a total mystery. Here, too, the supposition of the numerical being of the world would provide a way out.

At this point, however, the limits of this doctrine make a striking appearance. It can be shown, and was, in fact, shown by the very author of the doctrine, that there are objects in the world which, though they are evidently in "some sort of relation" to one another, yet, scanda lously, have no numerical logos --- for instance the side of a square and its diagonal.[25)] It is, of course, the matter of Def. 5, Book V of the Elements to justify such irrational ratios --- ἄλογοι λόγοι[26)], by indirectly defining a quantitative criterion by which the equality (sameness is no longer a perfectly meaningful term here) of such logoi may be recognized. The point in referring here to this development is this: at that very moment at which the logos looses that direct meaning described above, which is based on the supposition of the numerical character of the world, it becomes as indispensable as it becomes opaque --- a device for comparing quantitatively the relations of non-homogeneous things and happenings in the world, where the ralations and their terms are left quite dark. In other words: logos becomes subservient to analogia.[27)]

H. Music as Mathematics

Music, the "Muses' art" (μουσική) is not named in either of the Euclidian Scripta Musica - the Canon or the Introduction to Harmonics.[28] And quite rightly. For the Muses teach "beautiful song"[29] to the musician, and beautiful song is not the business of the "canonician" (κανονικός) who cuts the canon (κανονίζει), or --- they are the same[30] --- the "harmonician" (ἁρμονικός). "Harmonics", the Introduction says (1, 2), "is the theoretical as well as practical science (ἐπιστήμη) of the nature (τῆς φύσεως) of what is joined together (τοῦ ἡρμοσμένου).[31] As a student of what is well-ordered, well-fitted, bound fast, the harmonician makes a "ruler" or canon from which other instruments can be tuned.[32]

What is this "theoretical and practical science of what is (to use the Greek term) in harmony"?

Let us look at this study, which we will, after all, for certain reasons to appear later, call "music", though it is primarily neither about melody or song. When particular strings are plucked in succession (or together), there is heard a "wiry concord, that the ear confounds".[33] The ear is confounded, that is surprised, arrested, amazed --- delighted Those wires which produce such confounding concords turn out to have lengths which are to each other in the ratio of a small number to a small number. Those concords which seem to have affected the Greeks most have the number ratios 4:3, 3:2, 2:1. Pythagoras, all agree, discovered this.[34] The Pythagoreans consequently swore by the "tetraktys" ten (i.e., the sum 1+2+3+4), which "sums up" the numbers which make harmony,[35] thus putting their trust in the intelligible concord of this world rather than in the dark river of "hatred" (Στύξ) of the underworld, which was the great oath of the rest of the Greeks.

Pythagoras' discovery proclaims music as the very paradigm of science. For there are many things in the world which eventually turn out to be amenable to ratio, but which do not sufficiently confound us in admiration. For instance, both how and why the natural sub-lunary motions tend toward the earth turns out to be eminently capable of formulation in ratios, --- but it is a fact so often confounding in the worse sense and so little reminiscent of those heavently motions of which it turns out to be a special case, that its "mathematical" character long escaped attention, keeping the science of motion from soon coming about. But music, beyond being inseparably both "aisthetic" and rational (cf. Note 15), is also "aesthetic" according to our modern usage of the word;[36] that is to say, it draws attention to that in the world which seems to be made for the predilection of men. I mean, therefore, that it is a "paradigm of science" not in the modern sense of "descriptive" "physical" "science", of a technique for formulating rules of motion, but in the

Greek sense of knowledge (ἐπιστήμη) of the world, and therefore about a world which can be known, i.e., about a thing which is and which is orderly, a cosmos. (The great question is, of course, whethere there is, in truth, such a world and such a knowledge.)

But the cosmological possibilities of music are not exhausted by the fact that its sources are the aesthetic in αἴσθησις and the arithmetic in λόγος. The predominantly rational part of music (which however cannot quite escape from its sensible sources) is, to begin once again, called harmonics. This science, even (or even especially) when treated quite technically, deals with objects which have a strange likeness to the sources of being and the elements of the world. These likenesses, partly built into the science by the Pythagoreans, but partly in the matter itself, had best be simply and soberly enumerated:

First: That from which the scale is "put together" or compounded is the interval --- διάστημα, literally "distance" or "that which results when something stands apart". This word is a synonym for διάστασις, "distance", meaning "dimension" in our vocabulary.[37)]

The greatest interval is the diapason --- διὰ πασῶν, the distance "through all things". This octave interval is the perfect consonance, that is to say, it is heard as being the consonance of the same with the <u>same</u>, and yet it is not a consonance of identity, for the second bounding tone is not coincident with the antecedent but the distance of the whole away from it and completely other.

Furthermore, the bounding numbers of the diapason are 1 and 2, which immediately bring to mind the monad and dyad, namely those which are always said to be the beginnings of being.[38)] The ratio 2:1, accordingly, differs from the arithmetical ratio of equals (1:1, 2:2, etc.) whose part it plays in music, in being able to effect a difference, a growth, αὔξησις --- "growth" is the technical word for dimension --- when compounded, as the ratio 1:1 cannot: "and the ratio of equals will not grow (αὔξεται) when compounded" (Theon, p. 83, 24-25; p. 107, 15 f.). Also, since intervals are distinguished from ratios by being reversible (i.e., the intervals 2-1 and 1-2 are the identical consonance, i.e., there is no proper antecedent or consequent, except insofar as the interval is "taken" upward or downward, but the ratios double and half are different Theon, p. 81, 11-12), intervals share the non-sequential nature of spatial dimensions.

Second: A consonance, or concord, is called συμφωνία in Greek --- what is "together in voice". This word is, as in English, easily and naturally used to mean "agreement", e.g. in <u>Republic</u> 402 d 1-2: ". . . leading young people by means of a musical upbringing into sameness (ὁμοιότης)and friendship (φιλία) and concord (συμφωνία)". Now the bearer of consonance, sometimes called simply "the consonance", is the interval, a "distance", which is, after all, ordinarily a source of difference: ". . . these two were parted in strife" (διαστήτην ἐρίσαντε - Iliad I, 7). But musical consonances effect agreement out of difference and bind what is separate, so that "consonance has the greatest strength, being truth in speech, happiness in life, and in nature, harmony" (Theon, reporting the Pythagoreans, p. 47, 1-2).[39)]

Third: The possibility of harmonics rests on the attunement of the human soul to the arithmetical properties of the world, since what are to be the consonances depends on what I find agreeable. More than this, the very notion of a musical sound, a "voice" (φωνή , see Section D) as the ancients called it, includes the idea of its being an expression of soul: "Voice is ensouled noise (ψόφος ἔμψυχος)" (Aristotle, On the Soul, 429 b 5-6).[40)] This is, of course, where the Muses, as teache of beautiful song, enter into the science of harmonics, and give it its common name, music.

Fourth: Musical intervals, or perhaps rather their bounding tones, are not only heard but also seen and touched, namely in the length of the strings producing them --- for instance, geometrically, in the diagrams to Props. XIX and XX of the Canon, or bodily, in strings; Theon (p. 71) discusses these "visible" and tangible intervals in connection with the physical cutting of the canon.[41)]

Fifth: Musical sounds are closely connected with motions and meetings of bodies, as the preamble to the Canon says.[42] Thus musical sounds represent the world insofar as it appears to be the common source of appeals to different senses, and insofar as these appeals seem to come through orderly motions.

Sixth: The scale --- σύστημα, the "system" or the "construct", is a closely fitted whole made by compounding consonant intervals. This means that it is a. constructed by a technician with a tool, namely the harmonician with his canon - which is, in fact, his tool (i.e., he has a musical instrument for his tool, his ὄργανον "instrument", Porphyry p. 22, 27 f.; cf. Notes 49, 61). b. the method of construction is by compoundi (see Section G), and in particular by compounding consonances, for the tones of the Scale are "taken" precisely "through consonance" (διὰ συμφωνίας - Prop XVII), as the Canon is there to show.

Of the preceding items, what is humanly most serious, and most lost beyond recovery, is the perfect trust in the affinity of soul and number. It is this affinity which allows the Pythagoreans to see the world not only as a ζῷον, "live thing" (see Note 5), but as an ἔμψυχον, a "thing ensouled".[43] And it is only if it is a ζῷον ἔμψυχον that the city "which is larger than a man" but, like a man - capable of excellence, can ever come to life, can be "put in motion".[44]

The world of the Pythagoreans is constituted to support this trust that the world, cities, and men are all alike "cosmic"; the (spurious) work which the Timaeus is supposed to plagiarize (!), is significantly, called: "On the Cosmos and Nature of the Soul". This is, I think, the chief thing to say about them --- for it is what distinguishes them from certain adherents of modern science who feel that the Pythagorean "arithmetization" of the world pre-figures mathematical physics and arithmeticized mathematics (i.e., analysis)[45]. I will try simply to describe briefly how this Pythagorean world is put together out of the musical items listed above.[46]

"Pythagoras was the first to name the container of the whole (τῆν τῶν ὅλων περιοχήν) cosmos from the order (τάξις) in it"; (Diels I, 105, 24-25, literally "container of wholes", i.e., a κόσμος κόσμων; περιέχω is the verb used of the sounds which "contain" the intervals.) The Pythagorean world is therefore an ornament (κόσμος, "adornment"), but what is responsible for its ornamental appearance --- that is to say, both for its making an appearance at all and for its being delightful and "Musical", is the orderliness of its arrangement.

This orderly arrangement has several aspects.

a. It is an order of priority, in which some things come first and others second. This order makes understanding first possible, insofar as to understand means to find what came first, what the beginnings (ἀρχαί) were: "the thing first joined, the one . . ." (Philolaos, Diels I, 410, 11-12; also 13 f).

b. It is *arrangement simply*. It makes limits, distances, differences, "through which"[47] arise the many things which the world "contains". This is effected by that "one" and "two" which contain the distance "through all". To understand how they do this is to take the noblest road of inquiry, "not so hard to indicate, but overwhelmingly hard to use".[48]

c. It is an order of *being*. The same one and two and the rest of the numbers make things be, even in a bodily way: "number... makes things have the nature of bodies"(Philolaos, Diels I, 412, 1; also 454, 3). This is to be inferred from our grasping them, for did they not "have" number we would be able to think nothing and know nothing" (Philolaos, Diels I, 408, 6) --- to "take in" nothing. But if things "*have*" number they *are* number as Aristotle (*Metaphysics* 987 a 22-23) accuses the Pythagoreans of reasoning. (The *integrity* of these numbers, which make the being of things, is, of course, their most important characteristic; one might say that they put integrity, i.e., wholeness, into the world. It is therefore unthinkable that anything could be done to break, i.e., to fraction "the Pythagorean unit: τὴν μονάδα διαιρεῖσθαι ὅπερ ἀδύνατον," to divide the unit --- which is "impossible", as is said in Prop. III of the *Canon*).

d. It is a bond (σύνδεσμος - Timaeus, passim). "Since these beginnings underlay (ἀρχαὶ ὑπῆρχον - literally either "were present" or "governed from beneath"), being unlike and of different family, it would indeed have been impossible to make a cosmos (κοσμηθῆναι) with these, had not harmony also come about, in whatever way it did come about" (Philolaos, Diels I, 409, 2 f.). At this point the "ornamental" or delightful aspect,* comes in, namely by way of the logos: "We delight in consonance because it is a mixture of opposites having a logos with one another. For logos is an order (τάξις) which is sweet by nature" (Aristotle, Problems, XIX, 38). The logos makes all things into "relatives"; as ratio it allows agreement and consonance to arise over distances or intervals. But the bond is not only in having relations, but in having "fit" relations, in making a "fitted thing" (ἡρμοσμένον), which, in harmonic terms (Introduction to Harmonics, p. 186, 2-3), is "compounded from sounds and intervals having some order (τάξις)". is, in short, a "system". This is the great visible, and as the Pythagoreans thought, audible, heavenly order --- heliocentric, marvellously enough (see Note 42 a) --- which comes to be know to us through the different senses --- as a consequence "the body is cherished by the soul, because without it it cannot use the senses" (Philolaos, Diels I, 419, 3-4). But "within the soul, [number] fits all things in with sensation (ἁρμόζων

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* of the κόσμος , a rather mysterious and marvelous aspect,

αἰσθήσει) and thus makes them known (γνωστά) to and address (προσήγορα) one another according to the nature of gnomom" (Philolaos, Diels I, 411, 18 f.; the gnomom is to astronomy what the canon is to music, the special measuring stick of that science). The "harmonic" bond then, is what joins not only the world to itself but the soul to the world. This is why it is the Muses who tell the beginnings of the world. (Hesiod, Theogony, 116 f.). Those who receive their song have received a "sacred gift" (ibid, 96), identical, I think, with that "gift of the gods" - knowledge of the beginning of being - which Socrates received from the Pythagoreans (Philebus 16 c 5).

(One last aspect, belonging to the very word "order": an order is usually "given" or made by someone. The mythical, albeit prosaic, name of this maker of order is "the Artisan" [δημιουργός], literally, "public worker")[49]. His tool, by which he makes the cosmos --- as we in turn make it from the cosmos --- is the joiner's tool, the canon.[50])

What remains of this extravagance? Pythagoras himself discovered its most world-shaking difficulty --- that there are things in the world not numerable in relation to one another. In fact it is thought[51] that he came across this "scandal" in the course of his discovery of compounds and compounding, namely when attempting to divide the diapason equally, which is, in numbers, the problem of finding a mean between 1 and 2, or of finding $\sqrt{2}$, an irrational. But beyond this difficulty within mathematics itself, the whole thesis of number as the being of things is untenable. The arguments are in Aristotle --- no one could remain unconvinced (Metaphysics, 1092 b 2 f.). Even the heavenly order turns out to be more amenable to a treatment which does not concentrate on the harmonic arrangement of the planets. All in all, the notion of a directly "learnable", i.e., literally "mathematical" world order seems to be discredited --- both on account of the discoveries of the new science, and also because of the nature of aisthesis itself and of its objects.[52)]

Two aspects of this cosmology seem to me yet to retain their force. One is the science of harmonics itself, of which a little more below. The other is precisely that "naive" view with which we began. Aristotle calls it "too simple" and ἐπιπολαίως, "superficial" (Metaphysics 987 a 22-23). It consists in supposing that that "to which the term spoken of first applies, is the being of the thing" (ibid. 24 f.). It seems to me, however --- and Aristotle affirms this in his very writing of a history of earlier views --- that it is not possible to go beneath the surface before having been on it.[53)] I mean that the "naive" collapse of "that-through-which" into the thing itself, of the instrument into the artifact, of the knowable into the known, of the sense "data" into "reasons", of the body into the soul --- in short of the world into its being, is the only beginning for an inquiry which is to go into the depth of things. For it establishes that order of desire which alone can make the inquiry move. And therefore Socrates speaks of the initiatiors of this inquiry as "the ancients, who were stronger than we and lived nearer to the gods" (Philebus, 16 c 7-8).

We began by asking what harmonics or mathematical music was. We have seen that it was once conceived as the theoretical and practical science of the world, but that this turned out to be impossible. What is left?

The passage Republic 530 e f. is about harmonics. Musicians, Socrates says, labor over the endless task of determining the hearable consonances and deciding what the "dense" indivisible interval is (cf. Prop. XVIII and F. Notes). There follows a glorious parody of the way these people talk[54] those who "put their ears ahead of their thought" (ὦτα τοῦ νοῦ [οὖς : νοῦς] προστησάμενοι)". They speak of "beatings" (a play on πληγαί --- the cause of sound, see the preamble of the Canon) and subsequent "accusations and denials and boastings" (a take-off on the notion of musical "voice", and of consonance and dissonance as assent and dissent).

In short, these musicians get caught in futile attempts to settle questions concerning precisely those matters which become the assumptions of the Canon (cf. Section G), namely 1. what intervals are to be called consonances and 2. what the least interval is to be. Once having agreed on these, Socrates might have gone on, they pay the penalty for having started at the wrong end --- that is, with the ear's data rather than with a "problem" (Republic 531 c 2) in mathematical construction unburdened by the requirements of sense. For they discover a certain intractable unreasonableness in their mathematics, an insufficiency --- which is presented without a word of comment in Prop. IX of the Canon --- namely that once the tone interval has been "obtained through consonance" it is not possible to "obtain" the diapason by compounding all six tones which lie within it; instead the resulting ratio is an interval which exceeds the diapason by a piece called the "Pythagorean comma" --- "Pythagorean" because characteristically, they themselves discovered the difficulty.[55] A scale "obtained through consonance" is therefore not perfectly fitted. Socrates wants this whole enterprise to cut loose from its foundations in the senses, and to investigate rather what numbers are consonant and why (531 c 3-4)

What would the content of this purified harmonics be? Everything in this section has been directed toward showing that the Canon, at least, is not about such a harmonics,[56]* What then, might we imagine that such a work would contain? I think that the proper content would be precisely that theory of compounding so conspicuously omitted by Euclid (see Section G). Although I have found no ancient text that says so, there is a certain plausibility in this. Nicomachus (Introduction to Arithmetic, I, 3, 1, ed. D'Ooge, p. 184) classifies the science of "how much" (τὸ πόσον - quantity) into arithmetic, which deals with number "by itself" (καθ' αὑτό), and music which deals with number in relation to others (πρὸς ἄλλο). This latter science is called by Plato "logistic" (λογιστική)[58], the science of ratios. Now as music is the wider term, of which harmonics

*nor does there seem to have been such a work[57].

is a part, so compounding is a part of the wider theory of proportions, namely the part dealing with the "fitting together" of systems.[59)]

But why did this pure harmonics never develop? That question seems to me to open a wide vista for speculation. Possibly, to improvise, those who considered the matter saw that if they went ahead they would presently subvert from the inside the very character of that mathematics which is "useful in the search for the noble and the good" (Republic 531 c 6-7). For I have tried to show (Section G) that this operation of compounding has something irremovably "physical" about it; that is, it is a perspicuous process only when "something" i.e., an interval is compounded, while it passes the "naive" understanding what a putting together of relations might be. Therefore any formulization of this process must have gone the way of the spurious fifth definition of Book VI of the Elements; that is, the ratios would have been turned into a new kind of number, a kind of "relation-made-object", so as to be multiplicable.[60)] Then the decline of the logos referred to at the end of Section G would have been formally completed in antiquity.[61)] For, as we saw, the ratio in numbers, independent of and prior to proposition[62)], and sayable without reservation had to give way to a ratio of irrationals, an unsayable ratio, and a ratio meaningful only in its relation of sameness with other ratios in that proportion. And as ratio had once to give way, so now proportion itself would have ceased to be in command, namely in command of a collection of same ratios and would have become simply a relation of variables, so that "to compound" would have been to establish "joint variations".[63] Such a harmonics, would have turned out to be, albeit no longer "Musical", yet apt to the study of the physical world --- and insofar as this development, which is realized in modern times, is already contained in Pythagoras' mathematical music, he has left no small bequest.[64]

Notes to G. and H.

G.

1. Ptolemy, Harmonics, ed. Düring, Göteborg, 1930, p. 19 f., in a chapter entitled "That the Arixtoxenians (i.e., Pythagoreans, see Section D for Aristoxenus on Interval) unjustifiably make intervals and not sounds the measure of consonances", says: ". . . but although they [sounds] are, as it were, bodiless, they compare only their intervals... which are, after all, between bodies." The Pythagoreans are, it appears, accused of 1. taking the con-sonance for the sounds, or more generally what is between (μεταξύ) for the things themselves 2. mixing the realms of sense, i.e., sight and sound. Speaking as a technician, Ptolemy of course ignores the deep reasons for this.

2. Certain chief elements of this theory were evidently known in Plato's time, cf. Becker, "Eudoxos-Studien III", Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik, III 1936, p. 244.

3. Euclid uses μέρος, (V Def. 1; VII, Defs, 3, 4) but μορίον occurs in the name for superparticular numbers, ἐπιμόριος.

4. This, of course, has nothing to do with when the Canon was actually written (see Section C). The word "scandal" is Tannery's. I found it in Stenzel, Zahl und Gestalt, 1959, p. 88, together with a wealth of other references both ancient and modern. For the Pythagorean discovery of the irrational see The Thirteen Books of Euclid's Elements, ed. Heath, III, p. 1 f.

5. This definition of world --- before it becomes a cosmos --- does not seem to me to be too abrupt, especially not for the Pythagorean context, see Section H. The equivalence : world = place of life is made by

the Pythagoreans in a most literal way, for they view the world as full of life, as alive, as an "animal" an ἔμψυχον (Diels, I, 449, 7). Whatever extravagances, high or low (e.g. Diels, I, 449 f.; Timaeus 33 b f.; Rabelais, Pantagruel II, 32) may accompany this view, and leaving aside also cosmological theories, it seems to me to have a serious center in human experience, which is expressed in the identification of world and life in our ordinary speech (e.g. seeing the world; seeing life) --- whatever else it is or is not, the world is what we "count on" to carry us on and in which we carry on our life, once we have left the world of the womb.

6. Of course, I do not mean to imply that the word was first used in this way and that all the other meanings developed one after the other in time, since I do not know what, in fact happened. I mean only that certain simple, primary characteristics of the meaning of λέγω in all its uses are preserved in Homer. In short, I do not mean that those who spoke earlier spoke more wisely, and certainly not that the language is itself independently wise, but only that those who came first had less to become oblivious of.

7. Much has been written about this, most of it in inaccessable places and languages. I am unfamiliar with the mass of it. But I will be glad to help people who wish to read on this to certain useful attempts to understand, which I know of.

8. "Naive": from the Latin nativum, "natural", from birth", "unartificial".

9. Aristotle, On the Parts of Animals, 687 a 7 f.: "Anaxagoras said it was because of his having hands that man is the wisest (φρονιμώτατος) of animals"; cf. also, On the Soul 432 a 1 f. Compare with this the fact that the "wisest thing" (σοφώτατον) of the Pythagoreans is number (Theon, p. 99, 16-17).

10. In the best case the difference between the human and the mathematical relation vanishes, cf. Aristotle, Nicomachaean Ethics, 1166 a 31-32, where the good man "is related (ἔχειν) to his friend (πρὸς τὸν φίλον) as to himself". It seems to me a good investigation to see exactly how Aristotle's mathematization of human affairs (e.g. ibid., 1106 a 27 f.: excellence as "mean"; ibid., 1132 b 21 f. : justice as proportion) differs from the Pythagorean notion of number-virtue, which he clearly thought extravagant (Magna Moralia, 1182 a 11 f.). I must omit this whole Pythagorean doctrine here, which, in any case, contains too much of the sublime and the ridiculous at once, of the latter where it approaches its low perversion - numerology. One observation: for the purposes of the doctrine of

re-incarnation all things born with souls are said by the Pythagoreans to be "of like kind" (ὁμογενῆ , Diels I, 100, 40). But this is also the mathematical condition for being in ratio (Euclid V, Def. 3). In words, the difference is that between ἀναλογία and ἀριθμός.

11. The very phrase λόγος ἐν ἀριθμοῖς expresses the coincidence --- the logos is <u>in</u> the numbers. In the preamble it occurs in its technical sense of number-ratio but the same phrase, when used in larger context about or by Pythagoreans means nothing necessarily technically mathematical, but rather that the thing is to be accounted for by numbers: e.g. "seeing the <u>affections and reasons</u> of harmonies in numbers", Aristotle, <u>Metaphysics</u> 985 b 32-33; Diogenes Laertius, VIII, 29: οἱ λόγοι τῆς ζωῆς, "the logoi of life; 30: οἱ λόγοι τῆς ψυχῆς ,"... of soul." In these examples number ratios are, in fact, what will deliver the accoun but this is, purposely, not explicit in these phrases; see H. Diels, <u>Fragmente der Vorsokratiker</u>, Berlin 1954, III, p. 258. The opposite point of view of that which makes logos and arithmos coincide is expressed by Aristotle: "For logos is being, and arithmos material" (<u>Metaphysics</u>, 1092 b 18-19).

12. Aristotle, <u>On the Heavens</u>, 275 a 13; <u>Physics</u>, 215 b 13; also Aristoxenus, quoted in Diels, I, p. 451, 24: "All numbers have a ratio with one another". On the primacy of numerical ratio, see <u>Metaphysics</u>, 1020 b 31: "The first meaning of relations is that according to number".

13. Euclid's <u>Phaenomena et Scripta Musica</u>, ed. Menge, <u>Introductio Harmonica</u> p. 187, 20; p. 200, 2-3 (on irrational intervals); Aristoxenus, I, 8. There are also perfectly rational elements of voice --- aside from pitch, cf. <u>Philebus</u> 17 a 8 f.

14. <u>The Harmonics of Aristoxenus</u>, ed. H. Macran, Oxford, 1902, p. 236: κατὰ συμπάθειαν συνηχῆ , "sounds along in . . . fellow-feeling"; also, Theon, p. 51.

15. <u>Porphyrios, Kommentar zur Harmonienlehre des Ptolemaeus</u> 1932, ed. Düring, p. 92, 22 f., makes things simple by merely stating that most of the Pythagoreans who worked on the <u>Canon</u> say "interval" for "ratio", and he quotes Euclid's <u>Canon.</u> See on definitions of interval in general, Porphyry, p. 20 f., who also discusses and quotes authorities at great length on the question of aisthesis vs. logos in music (p. 22 f.) and particularly in respect to consonances (p. 94 f.). About the Pythagoreans he quotes Ptolemy: "Pythagoras and those who followed his teaching wanted to accept sense (αἴσθησις)

as a guide (ὁδηγός) for logos in the beginning, to provide it, as it were, with some kindling (ζώπυρά τινα), but the logos which arose from this they pursued, when it became separated from sense, by itself . . . " (p. 23, 25 f.). * See also Theon Smyrnai, Exposito rerum mathematicarum ad legendum Platonem utilium (Exposition of things mathematical useful for reading Plato) ed. Hiller, p. 71, 4; ἐν αἰσθητοῖς καὶ ὁρατοῖς διαστήμασι": in sensible and seen intervals".

16. Theon, p. 100, 9 f.; cf. Section H.

17. Heath, op. cit. p. 189-190.

18. Except for Prop. III, which needs to speak of a numerical "mean" which will not fall (ἐμπέσεται) within the bounds of the superparticular intervals, and Prop. IX, in which the tone interval must be six times compounded and the mean intervals do not have special names, being not quite tones.

19. Heath, op. cit. pp. 133, 135.

20. In harmonics a distinction is made between a tone found κατὰ θέσιν ;"according to position" or καθ' αὑτήν, "by itself" and that found κατὰ τὴν δύναμιν , "according to the ability" to effect others or according to τὸ πρός τι πῶς ἔχον , " "what it is as having relations" (Porphyry, p. 165 f.). Strangely enough, Aristoxenus (p. 124, 15 f.) says: "By hearing (τῇ ἀκοῇ) we judge the magnitudes (μεγέθη)of intervals, but by thinking (τῇ διανοίᾳ) we contemplate their abilities to have an effect (δυνάμεις)." One would have thought that their size is an arithmetical matter and their interaction an aisthetic one. However, this difficulty illustrates the illusiveness of the character of an interval. cf. Notes to Prop. XIX, Section F.

21. ὅρος , "boundary", is used for a number in a ratio by Theon, Ptolemy, Porphyry; also Aristoxenus: διαστημάτων ὅροι (p. 140, 21).

22. Porphyry quotes Ptolemy: "The investigation concerning the canon --- of what is it composed? Of things presupposed out of the province of the musicians, and of things taken from the mathematicians" (p. 23, 10-12). This is in fact, not a bad description of mathematical physics.

23. Le Opere di Galileo Galileo, Florence 1933, VIII, p. 359 f.

24. The ancients used both "equality" and "likeness" or "sameness" of

* "saying that...the logos finds what is right (τὸ ὀρθόν) by itself and exposes sense (ἀπελέγχειν τὴν αἴσθησιν, p.23, 30-31".

ratios: "Proportion is equality of ratios" (Nic. Ethics, 1131 a 32, the term "same" is used in the passages that follow). "Proportion is likeness (ὁμοιότης) or sameness (ταυτότης) of several ratios" (Theon, p. 82, 6-7, who also uses "equality" later on). Equality is, of course, suggested by greater and smaller ratios (Euclid V, Def. 7) which can, however, occur only in magnitudes. Likeness or sameness points to the very widest ole of analogia as bonding the world (cf. Stenzel, p. 150 f., and ch. IX called "One through Analogia", which is full of references.).

25. See Heath, op. cit., III, p. 2 for incommensurability proof.

26. See Heath, op. cit., II pp. 120-121 for ancient attempts to explain what ratio could mean for non-homogeneous, and, by implication, incommensurable, magnitudes. The burden of meaning is put on the word which is supposed to explain sameness in such ratios, "ἀνανaίρεσις" "counter-removal", used by Aristotle. The only meaning I can attach to this word is that it prefigures Euclid V, Def. V by describing sameness of ratio in terms of sub-multiples (as Def. V describes it in multiples), i.e., remove the same parts of correspondents, say half, and the proportion still stands.

27. Theon (p. 82, 22 f.) quotes Eratosthenes as saying that "the source (ἀρχή) of proportion is ratio, which is the reason (αἰτία) for the coming to be (γένεσις) of all things which do not come to be in a disorderly way. For all proportion is of ratios, and the source of ratio is the equal (τὸ ἴσον)". He goes on to explain that all things must be broken up into their elements (στοιχεῖα), and "the equal cannot be broken into more ratios". He explains that, just as the monad cannot grow by multiplication, so also the ratio of equality (1:1) does not exist in harmonics. Instead, there is the double ratio of the diapason (1:2).

28. Except in a quotation from the poet Ion, Introduction, p. 216, 21.

29. Hesiod, Theogony, 23. The whole distinction I wish to make might be set out in the analogy --- music : theogony :: harmonics : cosmology.

30. Porphyry, p. 23, 8: " 'Canonicians' are the Pythagorean harmonicians".

31. Axistoxenus and Ptolemy both begin their Harmonics with the most narrowly technical definitions of harmony. It is no doubt the Pythagorean background of the Euclidian works which prevents this.

32. Porphyry, p. 23, 27 f. specially insists that the instrument which is called "the canon" is so named from being tuned according to the canon, and not the other way around.

33. It is a commonplace that the Greeks heard harmony in a sequential and not a simultaneous sounding of notes, but there are frequent indications that, at least for purposes of study, tones were sounded simultaneously (e.g. Aristoxenus, ed. Macran, p. 234). Indeed, what would be more likely?

The quotation is from Sonnet 108, 4. No one could more accurately describe the effect of consonances than Shakespeare. He had, by the way, a distinctly Pythagorean view of music (e.g. Merchant V, 1, 54 f.).

34. See Diels, III, under συμφωνία ; Theon, p. 56, 9 f..

35. See Diels, I, p. 402, 12: "the dekas is called trust (πίστις)...", p. 455, 9-10; Theon, p. 58, 13 f..

36. This word was first used as the name of the "science of the beautiful" by a follower of Wolff (Kant's teacher), Baumgarten, in a thesis published about 1735.

37. In Timaeus 36 f. the two words are used interchangeably, obviously to make this point. The tones which bound intervals are defined (Introduction to Harmonics, p. 86, 14-15) as a "breadthless place (τόπος ἀπλατής) receptive to becoming part of a construct (σύστημα i.e., scale)". The spatial metaphor is carried on throughout; e.g., θέσις , the "position" of tones in the scale.

38. See Stenzel, passim, for references to the ancient sources for one-two, monas-dyas, limit-unlimited, one-many, same-other.

39. Also Symposium 187 a, b: "For harmony is consonance, and consonance a certain agreement". The passage speaks on the nature of agreement, musical and otherwise, which, it is said, could never be of things that differ (ἐκ διαφερομένων) but music puts agreement into all these things, imbueing them with love for and like-mindedness with one another (ἔρωτα καὶ ὁμόνοιαν ἀλλήλων)".

40. Ptolemy (p. 10, 26) plays on the word ψόφος as disordered φωνή , and also (p. 10, 18 f.) on the speech-like, i.e., expressive, nature of consonances: "Therefore each sound alone is speechless (ἄλογος) for it is one and indifferent to itself, and speech belongs to those things which have relations, and first of all to two such".

41. cf. Republic 530 d 6 f.; Galileo, p. 107 (see Note 42 c).

42. a. The Pythagorean Archytas reports explicitly on the discovery that sound arises only when bodies strike each other (Diels, I, p. 432-433; Timaeus 67 b., 80 a f.), but these motions themselves are not mathematically

known. This is, of course, the source of the notion of this bodily world as a great stringed instrument, productive of harmony and receptive of soul; see Theon p. 138, 9 f. who reports this view as Pythagorean and quotes poetry by Alexander (born end of fourth century B.C.) in which the cosmos is called a lyre of which the sun is the middle string, the tone called the mese. Note that the Pythagorean world was heliocentric.

b. Motion enters into music again on a different level, namely in the musical artifact called melody (on its relation to harmonic see Aristoxenus, ed. Macran, p. 95), which moves with that paradigm of orderly motion called rhythm. This is the special competence of the Muses as the synchronizers of heavenly and human motions (Timaeus, 47 d, Epinomis 987 a 2 f.; by "human motions" are meant those stirrings of the soul "out" of its calm which we call e-motions). I have wholly omitted the consideration of melody, as the Scripta Musica do. It is a kind of improvisation on the objects of harmonics, and is therefore to harmonics somewhat as myths are to logoi, i.e., a sort of epi-cosmic activity.

c. Perhaps the best comment on this --- understandably meager --- place of motion and body in Pythagorean harmonics is Galileo's treatment of harmony, which is, quite properly, present in the first of the six days of his cosmogony, The Two New Sciences. Sagredo is dissatisfied by the reasons given "by those who have learnedly written on music" for considering the double and the sesquialter ratios as the "natural forms" of the diapason and the diapente. There are three ways of stating the ratios of these intervals, for there are three possible variable properties belonging to strings as bodies: their length, their tension, and their weight. To obtain the diapason: if the length is to be changed, the ratio is 2:1; if the tension is to be changed, the ratio is 1:4; if the weight is to be changed, the ratio is 4:1. So those "wise philosophers" chose their defining ratios quite arbitrarily when they took them from string lenghts. There is, however, a motion which gives unique ratios for consonances and incidentally explains "the reason for more or less perfect consonances or dissonances", namely the frequencies of air waves; these are simply such in consonances as to strike the ear with a certain clear correlation (Dover ed., p. 99, the musical terms are taken out of the Italian of the National Edition).

The new "reasons" for the arithmetical and psychological facts of harmonics are 1. measured motions (Salviati has discovered a way of counting frequencies, p. 101), 2. the connected properties of bodies expressed into joint variations, and 3. aesthetic simplicity.

43. The word "animal" glosses this deep and difficult distinction: "Everything lives which partakes of heat ...., all things do not,

however, have soul" (Diels I 449, 24).

44. Republic, 368 e 5; Timaeus 19 b 3 f.

45. cf. Whitehead, Science and the Modern World, ch. 2. This is not to say that, in a loose and personal way, the scientists themselves are not Pythagoreans in the sense described. I have certainly come across traces of this is the popular writings of De Broglie, Einstein and Heisenberg, but it seems to be mor e a matter of the feeling their science arouses in them than of the assumptions of the science itself.

46. This will, of course, be a most incomplete, though, I think, not a misleading sketch. The cosmology of the Pythagoreans has arithmological, geometric, astronomic, "physical" and "ethical" aspects, all of which fit in with the musical paradigm, being either antecedent or consequent to it. The advantages of music as a special study are precisely in its microcosmic, i.e., inclusive but small-scale nature (cf. Theon, p. 96, 10 f.": . . . the musical and geometric and arithmetical relations (λόγοι) of which the harmony of the whole consists".

The sources reporting on Pythagorean cosmology are collected in Greek in Diels I, 446 f. Of these, these passages in the Metaphysics are the chief ones:

985 b 24 f.: cosmological principles;
1092 b 8 f.: Pythagorean theory of number with a refutation.

Of the older Pythagorean texts, the chief ones are the fragments of Philolaos (ibid. p. 398 f.) and Archytas (ibid. p. 421 f.)

It is generally agreed that the Timaeus and the Epinomis utilize --- in whatever strange mode --- Pythagorean doctrines.

47. Pseudo- Alexander (p. 601, 3 f.) speaks of the "inbetween" and "distance" "through which" (δι' οὗ) things change into each other. Such things must be ὁμογενῆ .

48. Philebus 16 c 1-2. "There is no nobler road, nor ever will be, than that of which I am a lover", says Socrates (ibid. 16 b 23).* And he goes on to illustrate what he is saying by a consideration of the science of "voice" and then harmonics. Again music is the paradigm study for the being of the world.

49. The word "demiurge" is used in this context before the writing of the Timaeus, e.g. Heracleitus: "Logos is the demiurge of what there is" (Diels I, p. 145, 35); Philolaos: "father and demiurge [of the cosmos] " (ibid. p. 418, 11). Philolaos' expression shows the double

*in the beginning of a speech about this ordering of the world by the one and the many - he actually uses the word διακεκοσμένα (ibid. d 1).

nature of the Pythagorean cosmos: as living thing of living things it has a father; as artifact of artifacts it has an artisan (as against it taken φύσις, in which aspect it has a mover who works from the inside, *Metaphysics* 1070 a 7-8).

50. In ordinary life a canon tells what the right proportions are, and is used by architects and sculptors (e.g. Polycleitus' *Canon* of human proportions). It is, therefore, what the demiurge of the *Timaeus* must be using for the fitting of the whole. In another version "number is the tool for distinguishing (κριτικὸν ὄργανον) of the cosmourgic god" (Diels I, p. 109, 29). That would not be for the fitting part of the job, but for the constitution of the elements of the whole. And since the Pythagorean elements are numbers (*Metaphysics* 986 a 3), this is another example of how in the Pythagorean world tool and artifact coincide.

51. Stenzel, p. 101 (Tannery's idea).

52. This latter is, of course, both an older and a far more weighty difficulty (*Phaedo* 65 a 9 f.; 96 a 6 f.).

53. Those who have heard Mr. Strauss lecture will find this metaphor familiar. I have also heard reasons why it is not a metaphor.

54. I cannot help remarking that these tongue-in-cheek disquisitions of Socrates, be they on genealogy, local mythology, or technical subjects, all show something about him that we tend to forget altogether --- how many things he knew.

55. cf. The *Freshman Physics Manual*, p. 31 f. It is said there that the comma represents that residue of irremovable irrationality called "tragic necessity"; I cannot find the exact source for this but it makes good sense. But compare Plutarch's statement that "necessity is a-musical", Diels I, p. 358, 13. See Diels I, p. 105 on the comma.

56. It is, of course, the obvious assumption that it is an example of "pure" harmonics, see Tannery, p. 218 (who has reservations).

57. Mrs. Gustin has a great deal of information precisely on the problem of purely mathematical harmonics.

It might be objected that Nicomachus (*Introduction to Arithmetic*, II) and Theon (p. 73, 21 f.) present such a science, since they say they do. But nothing is demonstrated in their books (which contain no theorems), and they are consequently descriptions and not sciences.

Furthermore both say that this "harmonics in numbers" is useful for cosmological and physical studies which, they think are its object. In Theon (p. 17, 10-11), at least, this is pure and misplaced faith, accompanying the thoughtless Pythagoreanization of Plato.

58. See Gorgias, 451 b 5 f.

59. Aristoxenus, ed. Macran, p. 223. I gather that "being compounded of" was thought of as a counterpart, within the theory of proportion, to "being in proportion" from the fact that in the Elements the spurious definition of compound ratio was inserted in Book VI in the very place where the famous Definition V of proportion stands in Book V.

60. The movement

$$\underbrace{\text{arithmos} : \text{arithmos}}_{\text{logos}} \qquad \underbrace{\text{arithmos} : \text{arithmos}}_{\text{logos}}$$
$$\text{logos} \rightarrow \text{analogia} \leftarrow \text{logos}$$

is usually called "attaining a higher level of generalization", and looks like a process of unification, except that the elements to be unified drop out on the way.

61. See Mr. J. W. Smith's preface to his translation of Vieta's Introduction to the Analytic Art.

The cutting of the canon itself would have had special possibilities had musical consideration been dropped in favor of purely mathematical ones. For if the scale had been completely diatonic (in which case everything harmonic to the Greek ear would have been lost), the canon would have become a slide rule, in which the ordinal number of each tone in its position on the scale would have been that exponent of $\frac{9}{8}$ which was represented by that string length; i. e., the number of the tone third highest in pitch, 3, would have indicated that this tone's position on the canon marked the length $(\frac{9}{8})^3$, c. f. Notes to Prop. XII, Freshman Physics Manual p. 48. And so logarithms would have been invented.

62. Theon, p. 82, 22 f. quotes Eratosthenes: "The beginning (ἀρχή) of proportion is ratio . . ."; p. 111, 4 f.: "Thus a proportion, when dissolved, will be brought back to (ἀναλυθήσεται) that which it was before it (πρὸ αὐτῆς) and from which it was constructed".

63. On the relation of joint variation to proportion, see the manual called Variation and Units, p. 4 (available in the library).

64. This is an understatement. One great sign of the greatness of his tradition is the title of a book by the man who is regarded as the very model (Peirce, _Philosophical Writings_ ed. Buchler, II ii; Einstein, _Essays in Science,_ "Johannes Kepler") of the modern scientist: _Harmonice Mundi_ (Kepler, _Gesammelte Werke,_ VI). But perhaps what Aristotle says in _Metaphysics_ 987 a 29 f. counts for yet more.

Note on the Notes; in aid of those to whom this remains a concern --- this writing, to which there are 2^5 footnotes, comes in 2^3 sections, of which 2^1, the first and the last, are underscored; see Theon, p. 94, 10 f.

A NOTE ON APOLLONIUS' PARAMETER

A NOTE ON APPOLONIUS' PARAMETER
(Corrected Version)

Eva Brann

In Propositions 11-13 of the Conics of Apollonius the three conic sections are defined by means of a line called the "parameter" of the ordinates: ἡ παρ' ἣν δύνανται αἱ καταγόμεναι τεταγμένως, literally " [the line] to which [are applied] in square the lines drawn down ordinately". That is to say, a square is to be raised on the ordinate and its area transformed into a rectangle under the abscissa of the ordinate, i.e., under a segment of the diameter toward the vertex cut off by the foot of the ordinate. As the area is one falling alongside, or falling beyond (ὑπερβάλλον), or falling short (ἐλλεῖπον), the ordinate is known to belong to the section called a parabola or a hyperbola or an ellipse.

For those studying Apollonius for the first time it is sometimes difficult either to catch the implications of the way in which the parameter is derived, or to see that these derivations are really similar for all the sections. A conspectus of these derivations may therefore aid the understanding. *

* This conspectus is here given in partially modern notation for the sake of brevity. But this is not meant to be prejudicial to the standing of the ancient mathematical objects. Whether the "application of areas" is a kind of proto-algebra, a "geometrical algebra" or a bona fide geometric procedure based on a particular use made of the theory of proportion such as is made in Euclid VI (particularly Propositions 27 ff.); whether the compounding of ratios can really be justified within the ancient understanding of ratio; whether the sections themselves are really graphs of functions plotted in crypto-coordinates or plane figures made by passing a plane through a solid; -- these and similar matters should, I think, be discussed with

a mind as free of prejudice as possible.

Modern scholars tend toward the "algebraic" interpretation. For instance, O. Neugebauer in an article significantly entitled: "Apollonius-Studien. Studien zur Geschichte der antiken Algebra", Quellen und Studien, III, 3, 215 ff. (from which article the expressions below for s.t are taken) holds that the Conics contain a "latent algebraic component" and that the "material content" of Greek and earlier modern mathematics does not basically differ. The whole question of their difference therefore becomes for him a question of mere method or style (which he takes to be equivalent terms) and quite separable from the objects to which they are applied.

Descartes precedes Neugebauer in granting that the ancients may have been in some respects as good as modern. But his view of what this means is radically different. For he believes method to be no mere external matter of presentation, nor at all separable from its objects. Consequently, when he credits certain ancient mathematicians with knowing his own analytic method, which is the "true mathematics", he writes as follows (Regula IV): ". . .thus by the same illumination of the mind by which they [the ancients] saw that virtue is to be preferred to pleasure, and what is honest to what is useful, although they were ignorant of why this was so [sic!], they recognized true notions in philosophy as well as in mathematics, although they were not yet able to grasp these sciences through and through. Indeed, certain traces of this true mathematics seem to me to appear in Pappus and Diophantus who, though they were not of the earliest age, yet lived many centuries before our time. But I believe that it was then suppressed by these authors from a certain low cunning (perniciosa quadam astutia); for perhaps they feared, as we know many craftsmen to have done in respect to their inventions, that once it was commonly known, it would be held cheap because it was very easy and simple; and they preferred to exhibit for us in its place, as the effects of their art, certain fruitless truths, very cleverly demonstrated by deduction, so that we might admire them, but they did not teach us the art itself, which would have put an end to our admiration once and for all".

[The Pappus mentioned has given his name to the three-and four-line locus problem treated earlier by Apollonius (E. B. edition pp. 799 ff.). This locus problem has a sort of intermediate nature

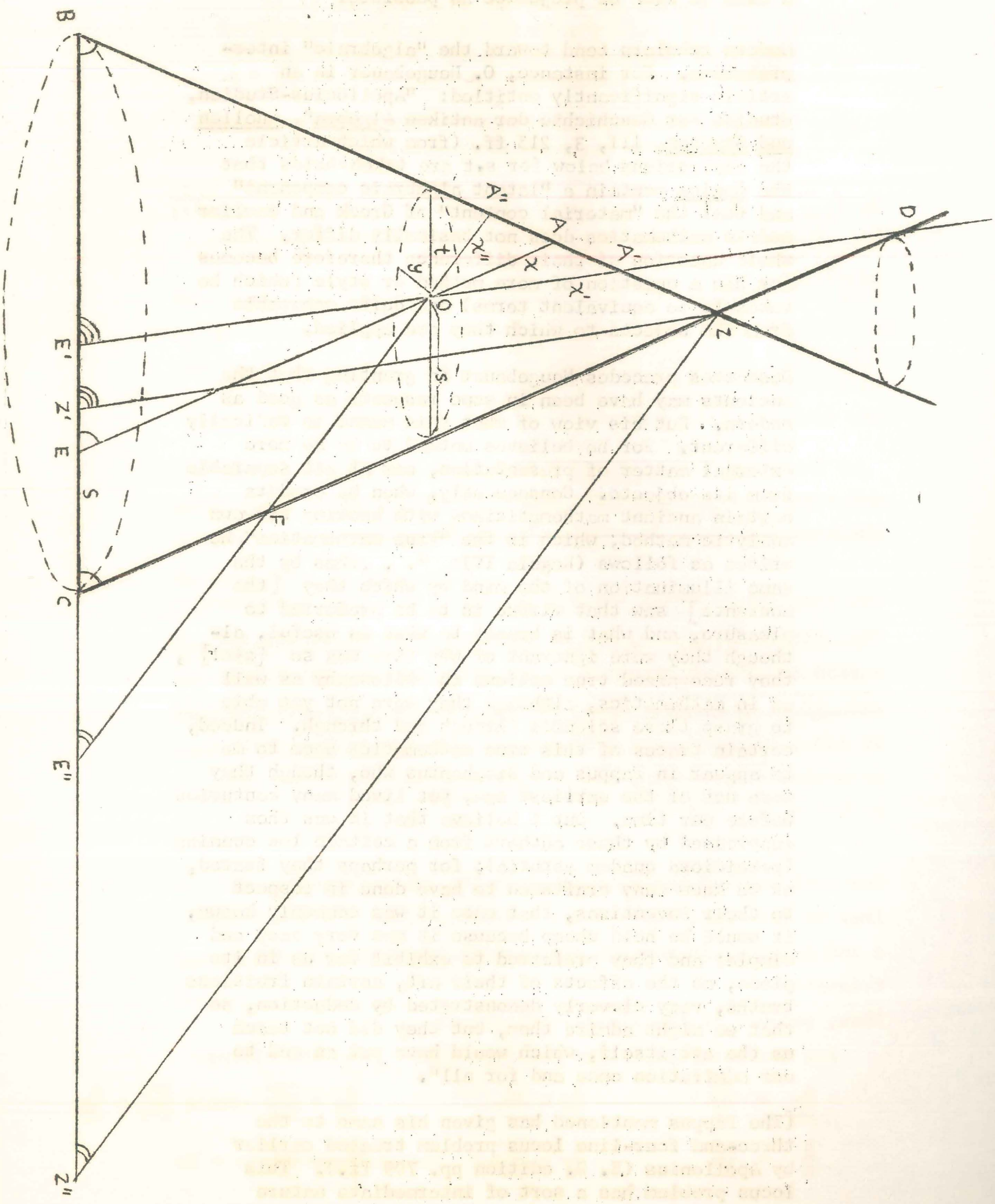

Figure I

between conics and analytic geometry, and it is indeed its general solution in the first book of Descartes' Geometry which marks the real beginning of that latter science.]

It is, of course, the object of this Note to uncover a little of the "method" of Apollonius.

In Fig. I let BZC be the axial triangle of a right cone, i.e., a cone with a base parallel to the plane of the generating circle.

Now pass a plane through the cone parallel to the base. This will produce a circle. Draw one of its ordinates y, cutting the diameter into segments s and t. By Euclid VI, 13

$$s \cdot t = y^2$$

and this holds wherever the foot O of the ordinate falls and wherever the plane is passed through the cone.

Now pass three planes through the cone, such that each cuts the circle in line y, and so that one is parallel to, the second is neither parallel to nor cuts, and the third cuts side AC of the cone. These planes will cut the axial triangle in lines AE, A'E', and A"E" and will produce respectively a parabola, a hyperbola, and an ellipse. The abscissas are marked x, x', and x".

Now each section can be related to the axial triangle, and thus to the cone, through its ordinate, since $y^2 = s \cdot t$ and s and t appear in several proportions involving the axial triangle.

Thus, because of the similar triangles:

for the parabola	for the hyperbola	for the ellipse
$\frac{s}{BC} = \frac{AZ}{BZ}$	$\frac{s}{DO} = \frac{CZ'}{ZZ'}$	$\frac{s}{OF} = \frac{CZ''}{ZZ''}$, since $\frac{E''C}{E''F} = \frac{s}{OF}$
$\frac{t}{x} = \frac{BC}{ZC}$	$\frac{t}{x'} = \frac{BZ'}{ZZ'}$	$\frac{t}{x''} = \frac{BZ''}{ZZ''}$

so that:

for the parabola

$$s = \frac{AZ}{BZ} \cdot BC$$

$$t = \frac{BC}{CZ} \cdot x$$

for the hyperbola

$$s = \frac{CZ'}{ZZ'} \cdot DO$$

$$t = \frac{BZ'}{ZZ'} \cdot x'$$

for the ellipse

$$s = \frac{CZ''}{ZZ''} \cdot OF$$

$$t = \frac{BZ''}{ZZ''} \cdot x''$$

and:

$$s \cdot t = \left(\frac{BC}{CZ} \cdot \frac{AZ}{BZ} \cdot BC\right) x \qquad s \cdot t = \left(\frac{BZ' \cdot CZ'}{ZZ'^2} \cdot DO\right) x' \qquad s \cdot t = \left(\frac{BZ'' \cdot CZ''}{ZZ''^2} \cdot OF\right) x''$$

Therefore for each section the square on the ordinate is equal to an area produced by the abscissa and a magnitude derived for that particular situation (i.e., angle at which the cone cuts the base) of the section in that particular cone.

For the parabola this magnitude is <u>constant</u> for <u>any</u> y and is, when taken as the length of a line, its <u>parameter</u> P, the "parameter" being a constant "measure laid alongside" of something to determine its character.

For the hyperbola this magnitude includes a changing factor: OD, the diameter plus the abscissa x', changes as 0 is taken differently, i.e. as the circle is moved up or down in the cone. Subtract x' from OD; then the parameter is:

$$P_h = \left[\frac{BZ' \cdot CZ'}{ZZ'^2} \cdot (OD - x')\right],$$

since OD - x' is a constant, namely the diameter A'D.

For the ellipse the magnitude within the parentheses will become constant for:

$$P_e = \left[\frac{BZ'' \cdot CZ''}{ZZ''^2} \cdot (OF + x'')\right],$$

where OF + x" is the diameter A"F.

The latter two sections, the so-called central conics, which have diameters, are also each associated with a figure or shape (εἶδος). Behind the figure lies a ratio (λόγος) which gives the same look (εἶδος) to all similar figures, namely the ratio of the diameter of the section to its parameter. (The parameter therefore has no proper position, and it certainly is not fixed into the cone like a nail.)

Here an algebraic expression will show very clearly how the figure helps to define the particular section. See also Figure II, below.

Multiply out and rewrite the expression for P_h:

$$P_h = \frac{BZ' \cdot CZ'}{ZZ'^2} \cdot OD - \frac{BZ' \cdot CZ'}{ZZ'^2} \cdot x'$$

Multiply by x':

$$P_h \cdot x' = \left(\frac{BZ' \cdot CZ'}{ZZ'^2} \cdot OD\right) x' - \left(\frac{BZ' \cdot CZ'}{ZZ'^2} \cdot x'\right) x'$$

But

$$\left(\frac{BZ' \cdot CZ'}{ZZ'^2} \cdot OD\right) x' = s \cdot t$$

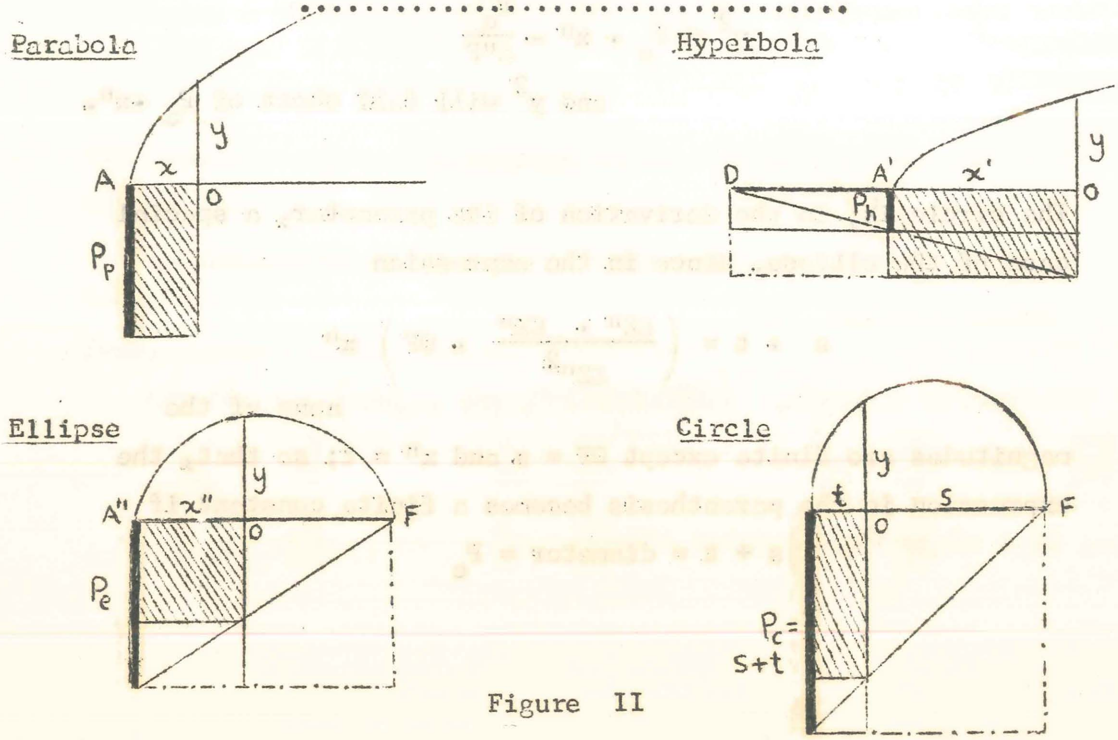

Figure II

So that:

$$P_h \cdot x' = s \cdot t - \left(\frac{BZ' \cdot CZ'}{ZZ'^2} \right) x'^2$$

Multiply the subtractive expression on the right by $\frac{OD - x'}{OD - x'}$

to get:

$$\left[\frac{\frac{BZ' \cdot CZ'}{ZZ'^2} \quad OD - x'}{OD - x'} \right]$$

This expression can be seen to be:

$$\frac{\text{parameter}}{A'D = \text{diameter}} \cdot \text{abscissa}^2 ,$$

so that after substituting and re-arranging

$$y^2 = P_h \cdot x' + \frac{P_h}{A'D} \cdot x^2$$

or the square on the ordinate as an area <u>falls</u> <u>beyond</u> or exceeds the area contained by the parameter and the abscissa by an area [equal to the square on the abscissa] which has <u>sides in the ratio of the parameter to the diameter; this is the</u> λόγος <u>of the</u> εἶδος .

The same thing can be done for the ellipse, in which case the expression will in the end be:

$$y^2 = P_e \cdot x'' - \frac{P_e}{A''F}$$

and y^2 will fall short of $P_e \cdot x''$.

The circle is, in the derivation of its parameter, a special case of the ellipse, since in the expression

$$s \cdot t = \left(\frac{BZ'' \cdot CZ''}{ZZ''^2} \cdot OF \right) x''$$

none of the magnitudes are finite except OF = s and x'' = t; so that, the expression in the parenthesis becomes a finite constant if

$$s + t = \text{diameter} = P_c$$

and

$$y^2 = P_c \cdot s - \frac{P_c}{\text{diam}} \cdot s^2 = P_c \cdot s - \frac{(s+t)^2}{(s+t)^2} \cdot s^2 = P_c \cdot s - s^2 ,$$

and thus the εἶδος of a circle is a square.

But whether it is really right to call the circle merely a "special case" of the ellipse, as has been done above, is an interesting question -- both in view of the general observation that special cases usually have low standing in modern mathematics, but also in view of the particular role the circle plays in the Conics in connection with the parameter. We are, as so often in Apollonius, by reason of his high cunning, left full of admiration for the device but quite in the dark concerning its discovery. How did he come to think of it? (Since it seems to be agreed that it was Apollonius who named the sections according to their falling alongside, beyond, or short of a parameter, there seems to be good reason to think that he too first derived parameters for all three sections.) Perhaps, we might answer, he found the parameter through considerations centered on the circle in some such manner:

The circle is in a way responsible for and prior to the cone, since it guides its generation, while it is also itself a section. It might therefore be possible to refer the other sections to it. Now the coordinate of the circle has a property well known from Euclid -- its square is equal to the product of the segments of the diameter, one of which may be considered as an abscissa. If then the sections are made to have this ordinate in common with the circle, several proportions arise by which their abscissas and, in the case of the central conics, some part or extension of their diameter are related both to the axial triangle of the cone and to the rectangle contained by those very segments of the circle. That is to say, the rectangle with an abscissa as one side and some magnitude derived from the axial triangle through these segments (and thus constant for that cone and that conic and hence "parametric"), will equal the square on the ordinate if there be added or subtracted a certain area. And on examination, this area proves to have the magnitude of the square on the abscissa and the look of a rectangle contained by the diameter and the parameter.