The Role of Mathematics in Advancing the Goals of a Liberal Education

by

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Mathematics appears in some form or other in the core or general education requirements of most colleges and universities, and a lot of energy is often expended regarding the specific form this requirement should take. For example, one thorny issue is just what level of mathematics should be required. Should it be sufficient merely to demonstrate proficiency in algebra thus basically requiring of students no more than they were supposed to learn in high school? Or should every student be required to take a bona fide college level course like calculus? Furthermore, what do we want the students to take away from the required course? Is our goal to demonstrate just how useful mathematics is in dealing with worldly matters and how indispensable it is with regard to understanding and interpreting real-world data and solving real-world problems?

With regard to the first question, namely, is proficiency at the high school level sufficient for meeting the core math requirement at a university, the answer has to be no. Certainly, we would hope to provide students with content and approaches which do more than merely replicate the high school experience. A student who has had four years of high school mathematics and who, therefore, has taken geometry, algebra, and in most cases a course covering standard precalculus topics should be offered an opportunity for intellectual growth which goes beyond a mere rehash of techniques studied in high school. Which leads to the question: just what type of intellectual growth are we talking about? Yes, mathematics is useful. Yes, it solves all sorts of technical problems. Yes, it enables us to analyze and interpret various kinds of real-world data. Yes, it enables us to model real-world situations and to understand and address a large variety of real-world issues. Is this the focus which we wish to emphasize in our core mathematics experience, or is there perhaps another direction we might take with regard to intellectual growth and development, particularly in a liberal arts institution?

I have repeated the phrase *real-world* because I think we do our students a disservice by focusing exclusively on the worldly aspects of mathematics. As Wordsworth reminds us, "The world is too much with us…" And he concludes:

> For this, for everything, we are out of tune;
> It moves us not.--Great God! I'd rather be
> A Pagan suckled in a creed outworn;
> So might I, standing on this pleasant lea,
> Have glimpses that would make me less forlorn;
> Have sight of Proteus rising from the sea;
> Or hear old Triton blow his wreathed horn. (Appelbaum 28)

It is that phrase, *glimpses that would make me less forlorn*, which is the key. At its heart, mathematics has nothing to do with the world. Its efficacy in understanding the world and in solving real-world problems is a fact, but this utility is a by-product rather than the central motivation behind the creation of mathematics. Mathematics offers glimpses - glimpses of the reality which underlies the physical realm, glimpses of the way in which our minds function and the ways in which we process ideas and the relationships between those ideas, glimpses of the role and limitations of language as vehicles for thought and the expression of ideas, glimpses of a beauty unlike any other leading to an aesthetic experience which rivals that of anything offered by the other fine arts.

What do we mean when we say that mathematics has nothing to do with the world? This requires some attention. First, what is mathematics? Definitions can be tricky things. A colleague once suggested to me that "mathematics is what mathematicians do." The English biologist T.H. Huxley offers a definition which is a bit more enlightening by noting that "[mathematics] is that [subject] which knows nothing of observation, nothing of experiment, nothing of induction, nothing of causation" (qtd. in Moritz 26). Notice that this definition describes mathematics by telling us what it is not. Observation, experiment, inductive reasoning (drawing general conclusions from specific cases) -- these play no role in mathematics. In a word, mathematics is not science. The truth claims of mathematics in no way depend on observations of the world. Rather, the reasoning involved is deductive (reasoning from the general to the specific) and depends only on our rational faculties. This leads to a very provocative definition formulated by the English mathematician J.J. Sylvester who, in contrasting mathematics and science, states that "the object of pure Physic is the unfolding of the laws of the intelligible world; the object of pure Mathematic that of unfolding the laws of human intelligence" (qtd. in Moritz 7). In other words, mathematical knowledge is self-knowledge. It tells us something of how we as human beings think, how our minds work, how we process ideas, and it demonstrates our natural propensity for finding connections between seemingly unrelated concepts.

At its root, mathematics deals with the most basic of human propensities, namely, collecting things and thereby forming a collection. Mathematics is about collections, and the mathematical term for this is set. A set is a collection of objects, and we can then think of the set as a single entity. Forming a single entity out of a disparate collection of objects is the essential mathematical act. *E Pluribus Unum*. Out of many, one. Every mathematical concept, including numbers, rests upon the notion of a set. There are two kinds of sets: finite sets and infinite sets, and infinite sets are central to just about every important mathematical theory. The simplest infinite set to grasp of the set of all positive whole numbers, what mathematicians call the natural numbers, N = {1, 2, 3, …}.

In a series of papers published in 1874, the German mathematician Georg Cantor undertook the first serious attempt to understand the nature of infinite sets. In particular, he wanted to understand whether or not we could meaningfully talk about the size of an infinite set and whether it is possible to compare the sizes two infinite sets. He asks whether all infinite sets have the same size, and, as a first step in addressing this question, he looks at the natural numbers and the seemingly larger set of all whole numbers (positive, negative and 0), known as the set of integers. Via a very simple and elegant argument, Cantor demonstrates that the set of integers is no larger than the set of natural numbers. It appears to be a larger set, but that is an illusion. In fact, these two sets have the same size.

Cantor then turns his attention to the set of rational numbers, namely, the set that consists of all whole numbers together with all fractions. This appears to be an incredibly huge set and one which certainly dwarfs the set of natural numbers. However, in one of most remarkable proofs in all of mathematics, Cantor shows that this apparent disparity in size is again an illusion and that the set of rational numbers has the same size as the natural numbers. This result and proof is one of the most famous and beautiful results in all of mathematics.

At this point, it is beginning to look as if the answer to Cantor's question is yes, all infinite sets have the same size. However, Cantor's most celebrated result, and the one which has deep implications for mathematics and philosophy, is that the answer is in fact no. He exhibits a set, namely the set of real numbers, which is so large that it no longer has the same size as the natural numbers. Infinite sets come in different sizes. There is not a single undifferentiated notion of infinity since some infinite sets are truly larger than others. This result completely changed theory of sets and, since sets lie at the heart of mathematics, all of mathematics as well. Prior to Cantor, going as far back as Aristotle, the very idea of an infinite set was rejected. The idea was rejected because it led to various unpleasant consequences, like the fact that a part can be equivalent to the whole. This is nonsense and so the very idea of the existence of infinite sets must be nonsense.

However for Cantor, infinite sets exist. The set of natural numbers exists as an actually, not just potentially, infinite set, and we can hold in our minds in a single instant of time the collection (i.e., set) of all natural numbers. Yes, this leads to certain startling consequences like the fact that a part can be equivalent to the whole. But the reason we are startled is because we are expecting infinite sets to behave the same as finite sets. This is an unreasonable expectation. Infinite sets are fundamentally different from finite sets and we can't expect them to behave in the same way. We are in a brave new world where a part can be equivalent to the whole and where some infinite sets are truly larger than others. This is not unlike the situation ushered in by the theories of general relativity and quantum mechanics where we are confronted with behavior which runs contrary to common sense and everyday experience.

Having proved that the set of real numbers is of a truly larger size than the natural numbers, Cantor wondered just how much larger it is. Is the set of real numbers the next largest set after the natural numbers or is there something in between? In other words, is there a set which fits in between the natural numbers and the real numbers in the sense that it is of a larger size than the natural numbers and of a smaller size than the real numbers? Cantor spent the rest of his life looking for the answer to this question. He felt strongly that the answer is no, but he was unable to prove this conjecture. This question, which is known as the Continuum Hypothesis, was singled out by the German mathematician David Hilbert at the International Congress of Mathematicians held in Paris in 1900 as the most important question which mathematicians must address in the new century (413).

And address it they did, although the eventual outcome of this endeavor was something which no one had expected or even considered. Results proved by Kurt Gödel in 1940 and Paul Cohen in 1963, taken together, show that the Continuum Hypothesis can neither be proved nor disproved. In other words, the Continuum Hypothesis is undecideable within the present axiomatic foundation of mathematics. Thus, while opening up whole new areas of human thought with his theory of infinite sets, Cantor’s work forces us to confront the fact that there are certain fundamental limitations on our capacity for understanding and that the answers to compelling foundational questions are simply unknowable by the human mind.

One of the weaknesses of many of the courses aimed at the general student is that all too often they morph into either a math appreciation course or a math history course thereby transforming the experience into something other than a course in mathematics. Even if this situation is avoided, another pitfall is that many such courses are survey courses organized around a multitude of topics. As such, each topic receives only superficial treatment and the student misses the essential experience of delving deeply into an important mathematical theory. I have sought to avoid both of these situations and have taught for many years a course organized around a few (at most three) selected topics in advanced mathematics.

These topics must be chosen carefully and, in my experience, need to meet several criteria. Each topic must be one which the students have not encountered before and must represent a truly important mathematical idea. Each topic should be modern in the sense that it originated relatively recently and is an active area of continuing research. Finally, each topic must be able to be presented to the general student without loss of mathematical rigor. This last point is particularly important since the whole point is to convey a meaningful sense of the mathematical experience. I have had considerable success in this course with topics chosen from number theory, group theory, and Cantor's theory of infinite sets.

Most of the students react very well to this course for two reasons. First, it is so unlike anything they expected from a mathematics course (this comment appears often on the course evaluations), and secondly they realize, especially from the discussion of infinite sets, where they are challenged to abandon certain of their preconceived ideas about reality, that mathematics has profound things to say about questions which have been raised for thousands of years by various important thinkers.

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