The Pentagon and the Heptagon

Recapitulation

Two weeks ago, we saw how Rafael Bombelli began to suspect that imaginary numbers might be meaningful as he worked on the cubic equation

\[ x^3 - 15x - 4 = 0 \]

Using the formula Cardano stole from Tartaglia, got

\[ x = \sqrt{2 + 11\sqrt{-1}} + \sqrt{2 - 11\sqrt{-1}} \]

which he was then able to solve by intuiting that

\[ 2 + 11\sqrt{-1} = (2 + \sqrt{-1})^3. \]

The second lecture described Caspar Wessel’s graphic presentation of the arithmetic of complex numbers. On the complex number plane --

(i) complex numbers can be expressed in polar coordinates by giving a distance (modulus) and an angle (argument);

(ii) multiplication of complex numbers amounts to

(a) multiplication of their distances (moduli) and

(b) adding their angles (arguments); and

(iii) the solutions of equations of the form \( x^n - 1 = 0 \), known as the “Roots of Unity,” appear graphically as the vertices of an equilateral n-gon in the unit circle on the complex plane.

Last week, we encountered the idea of “Constructible Numbers.” We showed that Euclid’s postulates allowed construction of lengths that correspond to the field of rational numbers, a collection of numbers that is closed under the operations of addition, subtraction, multiplication and division. In addition, Euclid’s postulates allow the construction of incommensurable magnitudes (which correspond to irrational numbers). However, Euclid’s postulates do not permit construction of all incommensurable magnitudes. We can only construct those that correspond to numbers that can be found in towers of finite quadratic field extensions, that is, field extensions that have a degree of \( 2^n \) over the rational
numbers. Plenty of numbers are not included. For example, \( \sqrt{2} \) is not constructible; neither are the non-algebraic (transcendental) numbers, which form an uncountable infinity far greater than the countable infinity of the algebraic numbers.

We now turn to an application of what we have seen so far: construction of a regular pentagon in a given circle.

The Lesser-Know Impossibility Problem

Ancient geometry knew several classical problems that seemed impossible; the three most famous were trisecting the angle, doubling the cube and squaring the circle. These were daunting challenges. No one had found how to accomplish any of them, but the ancients did not know whether they were really impossible or only difficulties awaiting clever solutions.

Only in the late 18th and 19th centuries did we learn that these three problems really are impossible, at least with the tools of Euclidean geometry.

In addition to these three, lesser-known but equally interesting problem is that of the heptagon, the regular seven-sided polygon. In book IV of the Elements, Euclid shows how to construct in a given circle an equilateral triangle (IV.2), a square (IV. 6), a pentagon (IV. 11) and a hexagon (IV. 15), regular figures with three, four, five and six sides. He then shows how to construct a regular 15-gon (IV. 16). Then he stops.

The reader might be expected to wonder, why? Why jump from 6 to 15? Euclid, in his customary laconic way, says nothing. Some of the figures he skips over were easily constructible. The octagon is easily made by bisecting the angles of the square. The 10-gon can be gotten similarly from the pentagon and the 12-gon, from the hexagon. But the orderliness of Euclid’s sequence really falls apart with the 7-gon. With what we learned last week, it is easy to see that the 7-gon is impossible to construct. It is obtained from the polynomial:

\[
x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0
\]

That sixth-degree polynomial is irreducible and, since its degree over the rationals is not a power of two, we can see right away that these complex roots are not constructible.

Did Euclid know that the 7-gon was impossible? He probably suspected it. He surely knew that he couldn’t do it, which is not quite the same thing.

Beyond the Impossible: the Unsuspected Possible

In 1796, at the age of 19, Carl Friedrich Gauss realized the impossibility of constructing the 7-gon; what is more, he realized at the same time that there are other polygons that can be constructed. Looking
only at those with a prime number of sides, in his book *Disquisitiones Arithmeticae*, he not only showed that the 17-gon *is* constructible, he showed how to do it. This is remarkable advance beyond what Euclid knew.

To help us get to Gauss’s result, it will be helpful to begin with a slightly simpler project: the algebraic construction of the pentagon.

**Euclid’s Construction of the Pentagon**

Of course, Euclid knew how to construct a regular pentagon in a given circle. To begin, let’s review how Euclid’s construction works.

**First, a Special Triangle**

He begins with construction of a very special triangle, one that is isosceles and whose base angles are both twice as big as its vertex angle.

\[
\begin{align*}
\theta \\
2\theta & \quad 2\theta \\
\end{align*}
\]

A little reflection shows why this triangle might be important to the construction of a regular pentagon: the three angles of the triangle total up to 180°, of course, but they also add up to five times the vertex angle. This triangle creates one angle that is one-fifth of 180°, and two that are one-fifth of 360°. If this triangle can be made, it will be the key to constructing the pentagon.

However, constructing this triangle is no simple matter.

To make it, Euclid recalls that back in book II, proposition 11, he had shown how to cut a line at a point so that the square on one portion of the line is equal to the rectangle contained by the whole line and the remaining portion of the line.
Digression

This kind of division is known as one into "mean and extreme ratio," sometimes also referred to as the "Golden Ratio." It has many cool features, including connection to Fibonacci numbers and logarithmic spirals, but we haven’t time to get into all these things right now.

If we take the whole AB to be “1” and the distance AC to be “x”, then finding this ratio can be understood as analogous to solving the equation:

\[ x^2 = (1 - x) \quad \text{or} \quad x^2 + x - 1 = 0 \]

whose solutions are:

\[ \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \]

You may note that these values are not rational, since they contains the square root of five. They are, of
course, constructible, which we know because (a) we are dealing with a quadratic extension of the rationals and (b) because Euclid in fact constructs one of them. (No surprise there.)

Returning to the Construction

Euclid takes a line divided in this way and, using one end as a center, draws a circle with the whole line as a radius:

He then makes a chord in the circle equal to the larger segment of the divided line:

He completes the triangle ABD, and joins CD:
Finally, he draws a circle that goes through points A, C and D:

Thanks to a proposition from earlier in Book III, he knows that when from a point outside a circle (like point B) a line cuts a circle (as line BCA), and another line is drawn to the circumference of the circle (as line BD), and when the rectangle on AB, AC is equal to the square on BD, then the line (BD) is tangent to the circle (ACD).

With that established, another proposition of Book III allows him to say that the angle CDB (angle 1) is equal to the angle CAD (angle 2):
Add angle CDA to both. Thus angles 2 + 3 are equal to angles 1 + 3. But because AB = AD (in the circle around A), angles 1 + 3 are equal to angle 5. So:

\[
\text{angle 5} = \text{angles 1} + \text{angle 3} = \text{angle 2} + \text{angle 3}
\]

and because of exterior angles in triangle CBD

\[
\text{angle 4} = \text{angle 2} + \text{angle 3}
\]

Therefore, triangle ABD is isosceles and line DB = line DC. And line DB = line AC.

Therefore, angle 3 = angle 2 = angle 1.

This, then, is the isosceles triangle with its base angles equal to twice the vertex angle.

**The Pentagon**

With the isosceles triangle having the base angles equal to the vertex angle now available, the rest is easy.
Simply bisect the arcs standing on the longer sides, which are each twice the arc on the shorter side. Now you have five equal sides and your pentagon is complete.

Join the vertices and you have not only a pentagon, but a pentangle (a regular five-pointed star).
This construction is completely rigorous and very clever. However, it offers no clues at all about how to pursue construction of other such prime-sided polygons, such as the 7-gon, the 11-gon, the 13-gon, etc.

Preliminary: the Pentagon

The algebraic construction of the pentagon amounts to finding the roots of the fifth degree cyclotomic polynomial. That is, we begin with the equation:

\[ x^5 = 1 \quad \text{or} \quad x^5 - 1 = 0. \]

The number 1 is evidently a solution to this equation. It is, in fact, the only rational solution. Therefore, the equation can be factored by removing the factor \((x - 1)\):

\[ x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1) = 0 \]

The Fifth Order Cyclotomic Polynomial

The second expression, \((x^4 + x^3 + x^2 + x + 1)\), is irreducible “over the rationals”; that is, it can’t be simplified by showing it to be the product of factors of lower degree among the rationals. We can be completely sure that this expression is irreducible because we know that the four roots of the polynomial \(x^4 + x^3 + x^2 + x + 1 = 0\) are complex with imaginary components. They are the four non-real fifth roots of unity.
But being irreducible over the rationals doesn’t mean that this thing can’t be factored in an extended field. In fact, it has been shown that every polynomial of $n^{\text{th}}$ degree can be factored into $n$ linear factors in the full complex number field. Our challenge is to find which factors need to be appended to the rationals in order to factor or “split” this fourth degree polynomial.

**Complex Conjugates**

We haven’t discussed complex conjugates, but this diagram presents the idea nicely. Notice that the complex roots of this polynomial appear as two pairs of complex numbers, symmetrically arranged above and below the real number axis. Root $\zeta^1$ is paired this way with root $\zeta^4$ and root $\zeta^2$ is paired with root $\zeta^3$. Being so arranged, these roots are written in this form:

$$a + bi \quad \text{and} \quad a - bi$$

The expressions are the same except for the positive and negative signs attached to the imaginary portions.

Complex conjugates have this handy feature: when a pair of complex conjugates are added, their sum is a real number. Also, when a pair of complex conjugates are multiplied together, their product is a real number.

This feature is handy because we are often looking for roots of polynomials whose coefficients are rational (or, in any case, do not involve imaginaries). Of course, you can construct an arbitrary polyno-
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This feature is handy because we are often looking for roots of polynomials whose coefficients are rational (or, in any case, do not involve imaginaries). Of course, you can construct an arbitrary polynomial with a random selection of complex roots:

\[
(x - (2 + 7i)) \ (x - (9 - 3i)) \ (x - (-15 + 4i)) = \ldots
\]

But if you multiply this trio out, you will have some imaginary coefficients.

\[
(x - (2 + 7i)) \ (x - (9 - 3i)) \ (x - (-15 + 4i)) = (813 + 699i) - (142 - 41i)x + (4 - 8i)x^2 + x^3
\]

In fact, the only way to eliminate the imaginary components from the expanded polynomial is if the coefficients occur in pairs of complex conjugates. That way, when the conjugates are multiplied, the imaginary components disappear.

Return to the Problem

To solve our fourth-degree cyclotomic polynomial:

\[
1 + x + x^2 + x^3 + x^4 = 0
\]

We will proceed in the usual, brash algebraic way: we will pretend that we already have the solutions. Then we’ll work to discover what they are. The Fundamental Theorem of Algebra tells us that this fourth degree equation has four solutions, which we will designate (as in the picture) \( \zeta^1, \ zeta^2, \ zeta^3 \) and \( \zeta^4 \). Roots \( \zeta^1 \) and \( \zeta^4 \) are one pair of complex conjugates; \( \zeta^2 \) and \( \zeta^3 \) are another pair.

**Two-Stage Solution**

Take the sums of \( \zeta^1, \ zeta^4 \) and of \( \zeta^2, \ zeta^3 \), like this:

\[
\eta_1 = \zeta^1 + \zeta^4 \\
\eta_2 = \zeta^2 + \zeta^3
\]

When added together, \( \eta_1 \) and \( \eta_2 \) sum up to -1 (because all the fifth roots of unity together sum to zero, and \( \eta_1 \) and \( \eta_2 \) include all the roots except \(+1 + 0i\):)

\[
\eta_1 + \eta_2 = \zeta^1 + \zeta^4 + \zeta^2 + \zeta^3 = -1
\]

Also, the product of \( \eta_1 \) and \( \eta_2 \) works out like this:

\[
(\zeta^1 + \zeta^4) (\zeta^2 + \zeta^3) = \zeta^3 + \zeta^4 + \zeta^6 + \zeta^7
\]

Restate this result with the exponents taken Modulo 5, because, on the unit circle in the complex plane, \( \zeta^5 = \zeta^0 = 1 \). Thus, we have
\[ \zeta^0 = \zeta^5 \zeta^1 = \zeta^1 \]
\[ \zeta^7 = \zeta^5 \zeta^2 = \zeta^2 \]

Substitute:
\[ \zeta^3 + \zeta^4 + \zeta^6 + \zeta^7 = \zeta^3 + \zeta^4 + \zeta^1 + \zeta^2 = -1 \]

Presto! We have the sum of the four non-real roots of the equations \( x^5 - 1 = 0 \). We know that these sum to -1.

**Building a Quadratic Equation for \( \eta_1, \eta_2 \)**

Great! We have two terms, \( \eta_1 \) and \( \eta_2 \). We don’t know what they are, but we do know that their sum is -1 and their product is also -1. Does that sound like a familiar situation? When we know that when we know the sum and product of two terms, we can construct a quadratic equation that has these terms as roots. In this case, we have:

\[ x^2 + x - 1 = 0 \]

whose roots are given by the quadratic formula:

\[ \eta_1 \text{ and } \eta_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1}{2} \left(-1 + \sqrt{5}\right) \text{ and } \frac{1}{2} \left(-1 - \sqrt{5}\right). \]

(The approximate values of these are 0.61803 and -1.61803.)

**Behold! Now It Factors!**

Remember that we said that the expression \( x^4 + x^3 + x^2 + x + 1 = 0 \) could not be factored over the rationals? Now it can be factored in an extended field when we append \( \frac{1}{2} \left(-1 + \sqrt{5}\right) \) -- or even if we just append \( \sqrt{5} \) -- to the rationals.

We have:

\[ (x - \zeta_1)(x - \zeta_4) = x^2 - (\zeta_1 + \zeta_4)x + \zeta_1 \zeta_4 \]

\[ (x - \zeta_1)(x - \zeta_4) = x^2 - (\zeta_1 + \zeta_4)x + \zeta_1 \zeta_4 = x^2 - \left(\frac{\zeta_1 + \zeta_4}{\eta_1}\right)x + 1 \]

This expression, \( x^2 - \left(\frac{\zeta_1 + \zeta_4}{\eta_1}\right)x + \zeta_1 \zeta_4 \), has coefficients that are in the extended field. The coefficient of \( x \) is the sum of the two roots \( \zeta_1 + \zeta_4 \); we don’t know them individually yet, but we know that they sum to \( \eta_1 \), which is in the extended field. The constant term is \( \zeta_1 \zeta_4 \); we know right away that the product of these two is 1 (product of their moduli, sum of their arguments).
The Four Singletons

Now look at the four roots individually:

$$\zeta^1, \zeta^2, \zeta^3, \zeta^4$$

We know how they sum in pairs:

$$\eta_1 = \zeta^1 + \zeta^4$$
$$\eta_2 = \zeta^2 + \zeta^3$$

We also know the products of the same pairs:

$$\zeta^1 \zeta^4 = \zeta^5 = 1$$
$$\zeta^2 \zeta^3 = \zeta^5 = 1$$

So we can make two more quadratic equations:

$$w^2 - \eta_1 w + 1 = 0$$ whose roots are $$\zeta^1$$ and $$\zeta^4$$, which are solved as $$w = \frac{\eta_1 \pm \sqrt{\eta_1^2 - 4}}{2}$$

$$y^2 - \eta_2 y + 1 = 0$$ whose roots are $$\zeta^2$$ and $$\zeta^3$$ which are solved as $$y = \frac{\eta_2 \pm \sqrt{\eta_2^2 - 4}}{2}$$

We now have enough information to solve for the four roots:

$$\zeta^1 = \frac{\eta_1 + \sqrt{\eta_1^2 - 4}}{2} = \frac{1}{2} \left(-1 + \sqrt{5}\right) + \sqrt{\left(\frac{1}{2} \left(-1 + \sqrt{5}\right)\right)^2 - 4} = 0.309017 + 0.951057 \, i$$

$$\zeta^4 = \frac{\eta_1 - \sqrt{\eta_1^2 - 4}}{2} = \frac{1}{2} \left(-1 - \sqrt{5}\right) - \sqrt{\left(\frac{1}{2} \left(-1 - \sqrt{5}\right)\right)^2 - 4} = 0.309017 - 0.951057 \, i$$

$$\zeta^2 = \frac{\eta_2 + \sqrt{\eta_2^2 - 4}}{2} = \frac{1}{2} \left(-1 - \sqrt{5}\right) + \sqrt{\left(\frac{1}{2} \left(-1 - \sqrt{5}\right)\right)^2 - 4} = -0.809017 + 0.587785 \, i$$

$$\zeta^3 = \frac{\eta_2 - \sqrt{\eta_2^2 - 4}}{2} = \frac{1}{2} \left(-1 + \sqrt{5}\right) - \sqrt{\left(\frac{1}{2} \left(-1 + \sqrt{5}\right)\right)^2 - 4} = -0.809017 - 0.587785 \, i$$

You can see that these solutions contain *radicals of radicals*. These expressions are not in the first extended field, but we can extend that field again (in a finite quadratic algebraic field extension) so that
it includes these four solutions.

These can be plotted on the complex plane:

Voila.

**More Important Than the Answer**

To summarize and review.

More important that getting the answer or than drawing the pentagon is to notice how the field extensions were built. Beginning with the rationals, which are all constructible, we first got the values for $\eta_1$ and $\eta_2$, which were the sums of $\zeta^4 + \zeta^4$ and $\zeta^2 + \zeta^2$ respectively, the two pairs of complex conjugates. These values were $\frac{1}{2} \left( -1 \pm \sqrt{5} \right)$, and thus required that we move into an extended field:

$$Q \rightarrow Q(\eta_{1,2})$$

This is a quadratic extension and is thus constructible. Then, getting the four roots themselves required another field extension. The four roots are $\frac{\eta_{1,2} \pm \sqrt{\eta_{1,2}^2 - 4}}{2}$, and each will require one more quadratic field extension.

$$Q \rightarrow Q(\eta_{1,2}) \rightarrow Q(\eta_{1,2}, \frac{\eta_{1,2} \pm \sqrt{\eta_{1,2}^2 - 4}}{2})$$

Sequences of quadratic field extensions are constructible.

Look again at what is happening here. At the outset, we knew that we had a fourth degree equation with all complex roots.
Look again at what is happening here. At the outset, we knew that we had a fourth degree equation with all complex roots.

\[1 + x + x^2 + x^3 + x^4 = (1 - \zeta_1) (1 - \zeta_2) (1 - \zeta_3) (1 - \zeta_4)\]

By segregating out the pairs of complex conjugates, we separated the factors on the right into two pairs:

\[1 + x + x^2 + x^3 + x^4 = \left\{ (x - \zeta_1) (x - \zeta_4) \right\} \times \left\{ (x - \zeta_2) (x - \zeta_3) \right\}\]

\[1 + x + x^2 + x^3 + x^4 = \left\{ (x^2 - (\zeta_4 + \zeta_1) x + \zeta_1 \zeta_4) \right\} \times \left\{ (x^2 - (\zeta_2 + \zeta_3) x + \zeta_2 \zeta_3) \right\}\]

\[1 + x + x^2 + x^3 + x^4 = \left\{ (x^2 - \eta_1 x + 1) \right\} \times \left\{ (x^2 - \eta_2 x + 1) \right\}\]

Is this interesting? Yes! If we confine ourselves to rational numbers, then our original equation could not be factored. If we admit \(\eta_1\) and \(\eta_2\), it could be factored into two factors. If we admit all the complex numbers -- really, we needed go so far; a finite field extension adding \(\frac{\eta_{1,2} \pm \sqrt{\eta_{1,2}^2 - 4}}{2}\) to the mix would be enough -- then it factors into four factors:

In \(\mathbb{Q}\)

\[1 + x + x^2 + x^3 + x^4\] is irreducible

In \(\mathbb{Q}(\eta_{1,2})\)

"" factors to \(\left\{ (x^2 - \eta_1 x + 1) \right\} \times \left\{ (x^2 - \eta_2 x + 1) \right\}\]

In \(\mathbb{Q}(\eta_{1,2}, \frac{\eta_{1,2} \pm \sqrt{\eta_{1,2}^2 - 4}}{2})\) "" factors to \((1 - \zeta_1) (1 - \zeta_2) (1 - \zeta_3) (1 - \zeta_4)\)

The procedure we have followed does exactly what is required for specifying constructible figures: it has made a sequence of finite field extensions, starting with the rationals, \(\mathbb{Q}\), and proceeding by quadratic field extensions until the polynomial with our desired points as roots is completely factored.

This stepwise factorization works for the pentagon because at each step it was possible to subdivide the roots into two groups, each of which could be shown to be a quadratic expression of the preceding group. That is not always possible.

**Conclusion**

We have seen here an application of the technique of algebraic decomposition. The equation we are trying to solve is broken into simpler and simpler parts as the field in which we operate is expanded step-by-step until we arrive at a final field, the “splitting field,” in which the polynomial can be completely decomposed into linear factors.
Unlike Euclid's way of working, this methodical procedure provides a framework for evaluating which polygons are constructible and which are not.

We will see this method play out on a larger stage next week with the construction of the heptadecagon.

Thank you.