Dynamical Chaos: Some Implications of a Recent Discovery

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My subject is a peculiar behavior of dynamical systems that has come to be recognized only during the last thirty years. In 1975 James Yorke christened this behavior chaos - perhaps a misnomer. Chaos is a Greek word that has no plural. Since Hesiod it has meant the nether abyss, the first state of the universe, or total disorder. The dynamical behavior that James Yorke called chaos is order and apparent randomness intertwined.

A dynamical system is - what? How about this? - a set of entities that interact so as to undergo a development. The entities could be planets or billiard balls; maybe cardiac muscle fibers or neurons. Maybe even bidders on the New York Stock Exchange; but let that go. In my illustrations this evening, the components will be chunks of matter, unbesouled.

To understand why dynamical chaos was recognized only recently takes a bit of mathematical background. Mathematically, dynamical systems are represented by differential equations. Differential equations are distinguished by containing instantaneous rates of change, velocities, say, or accelerations. Now empirically you can't measure an instantaneous rate of change, but only changes over finite intervals of time. A differential equation, therefore, is a hypothesis. To verify the hypothesis you must first solve the equation, or integrate it. That means, you must somehow eliminate the rate or rates of change, and obtain the value of the dependent variable - which is

what you are interested in - as a function of the independent variable, which is usually time. Thus you will have a relation you can check empirically.

Procedures for solving a good many differential equations were worked out in the 17th and 18th centuries. The solutions turn on what is called the fundamental theorem of the calculus, discovered by Newton and Leibniz. The procedures, like the differential equations, assume that time is continuous.

Not all differential equations are thus soluble - "analytically," as we say. It may be impossible to disentangle the dependent variable from its rate of change, or, if there is more than one dependent variable, to disentangle these variables from one another. Then you can't arrive at a formula giving each variable as a function of the time.

All linear differential equations are soluble. In these equations, the dependent variables and their rates of change occur only to the first power, and don't multiply one another. An equation in just two variables occurring only to the first power can be graphed as a straight line; hence the name linear.

But there are non-linear differential equations, in which some of the dependent variables, or their rates of change, are raised to powers, or multiply one another. Some of these equations are analytically insoluble.

In fact, most dynamical systems in the world can be modeled accurately only by non-linear differential equations, most of which are insoluble. It is dynamical systems modeled by such

differential equations, non-linear and insoluble, that exhibit the behavior that James Yorke called chaos. Here a small change in the independent variable can produce a sudden large change in a dependent variable. Such phenomena, we're told, flourish in nature; for instance, in the dripping of a faucet.

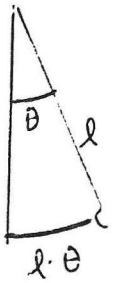
How can insoluble differential equations be studied? The chief way is by what is called numerical integration. This differs from the analytical integration I previously spoke of, which relies on the fundamental theorem of the calculus. In numerical integration the independent variable is not varied continuously; instead, it is increased by finite jumps. Starting with certain initial values of the variables, numerical integration assumes that the initial rate of change remains constant for some small, finite interval, say a second, and on that assumption computes the values of all the variables at the end of the second. Then with the new values it goes on to compute the values of all the variables at the end of the second interval. And so on. The procedure is not strictly accurate. But if the intervals are made small enough, it can give a good idea of what is going on; it can even, in many cases, be made to yield predictions as accurate as the observations.

The first large-scale numerical integration ever performed was carried out in 1758, to compute the return date of Halley's Comet. It took 6 months' work by three people, morning, noon, and night. Their final prediction was a month off, and even then they were lucky, because their computation contained some partially

compensating errors.

Recognition of the chaos named by James Yorke came only in the decades since 1960, with the development of highspeed electronic computers that could carry out numerical integrations no one had previously thought practical.

I am going now to illustrate this kind of dynamical chaos, and to talk about some of its characteristics. My interest in this subject arose because for some years I have been pursuing the question of how planetary astronomy became a precise predictive science, and since 1980 it has become apparent that planetary astronomy involves James Yorke's chaos.



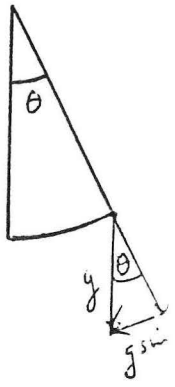
This chaos limits predictibility. Philosophers, I suspect, should learn about it. New perspectives open up if we recognize how widespread it is. More on this later.

I begin with the simple pendulum. It consists of a heavy bob, suspended by a weightless, inextensible thread - mathematical physicists love to invoke such things. If we draw the bob aside and let it go, it oscillates back and forth. On one side of the handout, I have derived its equation of motion, and shown it to be nonlinear. I'll repeat the argument now.

We measure θ, the departure of the thread from the vertical, in radians, defined as arc-length divided by radius. So the arc-length will be given by the radius-arm, or length of the thread, here l, times θ. In the science of dynamics as founded by Galileo and Newton, we are interested in accelerations, that is, rates of change of velocity; velocity itself being a rate of change of

position. Acceleration is thus a rate of change of a rate of change. In our case, we are interested in the acceleration of the bob, hence of its position as measured by the arc $l \cdot \theta$. But l is a constant; so the acceleration we are interested in is l times the acceleration of θ, which I write as theta with two over-dots, $\ddot{\theta}$.

The reason the bob accelerates is that it is pulled downward by gravity. The acceleration of gravity at a given spot on the Earth is a constant, which we call g. But the bob can't go straight down, with the acceleration g, because it is suspended by the thread. To find how much of g accelerates the bob along its path, we "resolve" g into components, one in line with the thread - this component merely tenses the thread - and the other component at right angles, along the path. In the handout I explain that the latter component is $g \cdot \sin \theta$. Our equation is then



$$\ddot{\theta} = \frac{g}{l} \cdot \sin \theta,$$

where, remember, the double over-dot means acceleration, radians per second per second.

I say that this equation is nonlinear. Sin θ is given by an infinite series - I won't prove this, please take it on faith:

$$\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

If only the first term were present, we would have a linear equation. But there are higher powers, going on forever.

Now it turns out that this nonlinear equation is soluble. I

won't write down the solution, which is somewhat complicated. It implies that the pendulum is not isochronous: wider-angled swings take a little longer. Galileo, gazing at the suspended lamps in the Cathedral of Pisa, guessed the pendulum was isochronous, and wanted so much to believe this, that he never made the simple experiments that would have shown the assumption false.

Of course, the simple pendulum is approximately isochronous, for small-angled swings. Suppose that θ is 6°, about 1/10 of a radian. In the series expansion for the sine, if the first term is 1/10, the second term is 1/6000. We can choose to ignore it, along with all the higher terms. That is called linearizing the equation. Mathematical physicists have been doing it for nearly 300 years, in order to obtain neat, soluble equations. The hope is always that the linearized equations give good enough approximations. And so they do, when the system is close enough to a stable equilibrium.

Suppose, then, we limit our simple pendulum to swings of 6° or less. The effects of nonlinearity will be present, but tiny. And now let us introduce a perturbation. When the word perturbation is used, we mean that there is some motion we can regard as fundamental, and some other disturbing motion that is superimposed. The Earth's motion is controlled primarily by the gravitational action of the Sun, but it is perturbed detectably by the Moon, Venus, Jupiter, Saturn, and so on.

Suppose the point of suspension of our pendulum is put into a small oscillation, in the very plane in which we first set the

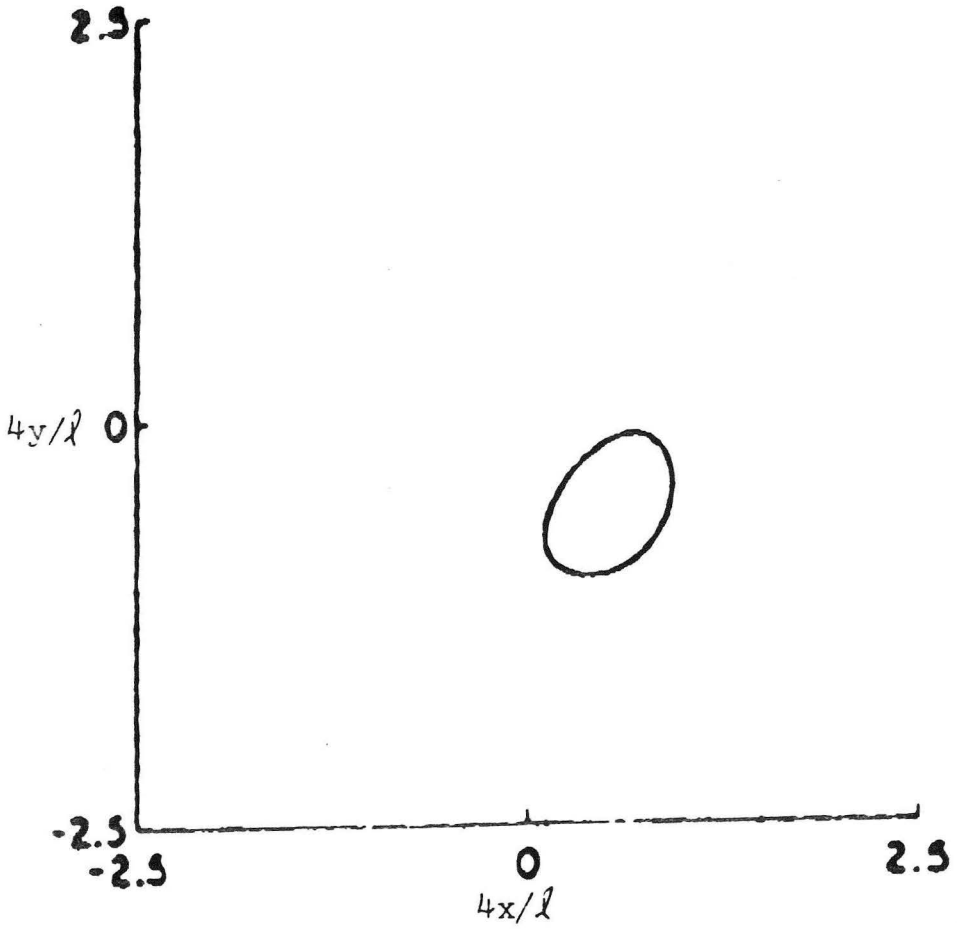
bob to oscillating. We could use a crank mechanism for this. Let the amplitude of this perturbing motion be a small fraction of the length of the pendulum. Let the period of the forcing motion be one we can vary; call it T. And suppose we set T to be somewhere near the period of our linearized simple pendulum, which I shall call T_0. T_0 is given by a formula some of you have learned, 2π times the square root of l over g.

$$T_0 = 2\pi\sqrt{\frac{l}{g}}.$$

The equation of motion for the new set-up, which I won't write down, is not soluble analytically; it won't yield a formula for θ as a function of time. But a numerical integration can be carried out. John Miles of UCSD did this in 1984, to determine the position of the bob each time the perturbing motion reaches the righthand end of its range. Starting from below T_0, he increased the period T of the perturbing motion. At a certain point, the motion of the bob, in its original direction of motion, which I'll call the x-direction, became unstable. But meanwhile there were two possible motions that were stable - motions that included a sideways component, a y-component. The motion of the bob made a gradual transition to one or the other of these stable motions.

When $T = 0.9924\ T_0$, the position of the bob each time the perturbing motion comes to the righthand limit of its excursion moves in this figure (Figure 1). You probably want to know what the whole motion of the bob is. It is in a slowly rotating ellipse with slowly varying axes.

But let me focus solely on the position of the bob each time

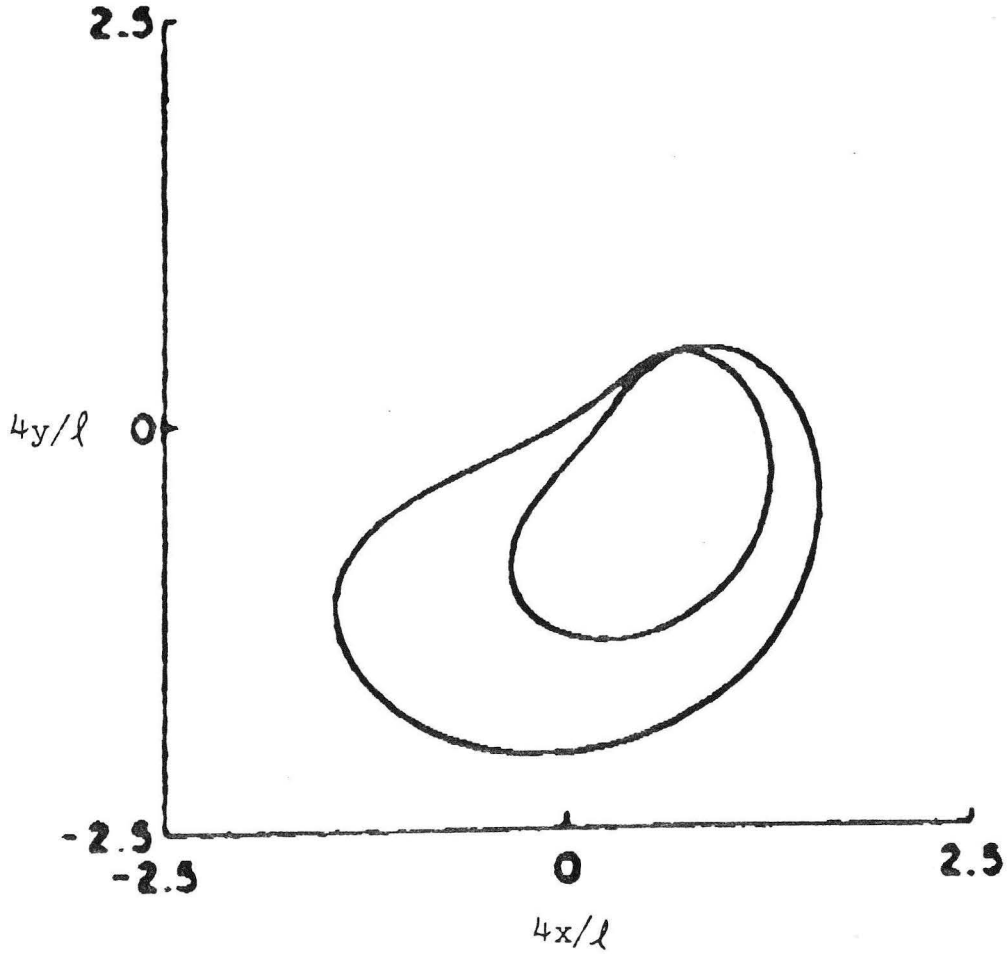


$T = 0.9924\ T_0$

the perturbing motion reaches the righthand end of its range. If T is 1.0150 T_0, our point moves in a doubled curve (Figure 2); what is called a bifurcation has occurred. A small quantitative change has produced a sharp qualitative change. Let the period be increased so that T is 1.0213 T_0 (Figure 3); another bifurcation has occurred. A cascade of further bifurcations occurs, as T is increased. When we reach T = 1.0225 T_0 (Figure 4), the figure appears smudged, with many paths close together. The pattern, if accumulated over a long enough time, appears to be symmetric with respect to the x-axis, although the bob may spend substantial intervals in either the top or bottom half of this pattern, transferring from one to the other at seemingly random times.

Random is - what? The word comes from old French randir, to run or gallop; the French knight, having donned his armor, and drunk certain flagons of wine, was hoisted by crane onto his horse and galloped about the field, doing random mayhem. The Anglo-Saxon, by contrast, wore bearskin, drank mead, and on the battlefield went berserk, a word meaning bearskin. To define random mathematically, is something else again. Perhaps we can say what it is not. If our pattern were two or more periodic motions superimposed, it would be what is called quasiperiodic; if the periodic motions were incommensurable, there would be no exact repetitions, but the motion would not be random or chaotic. But our motion doesn't look quasi-periodic, with those sudden shifts from one part of the pattern to another.

Let's turn to an actual physical experiment. Al Toft,



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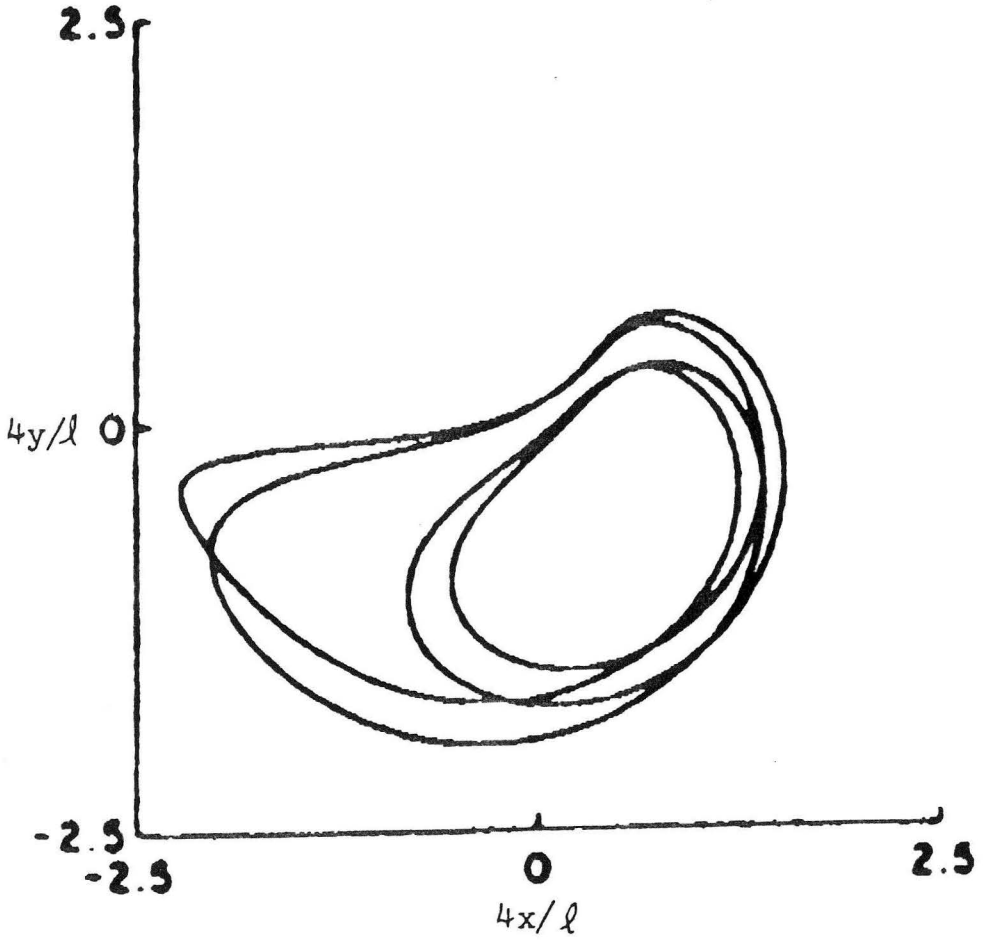

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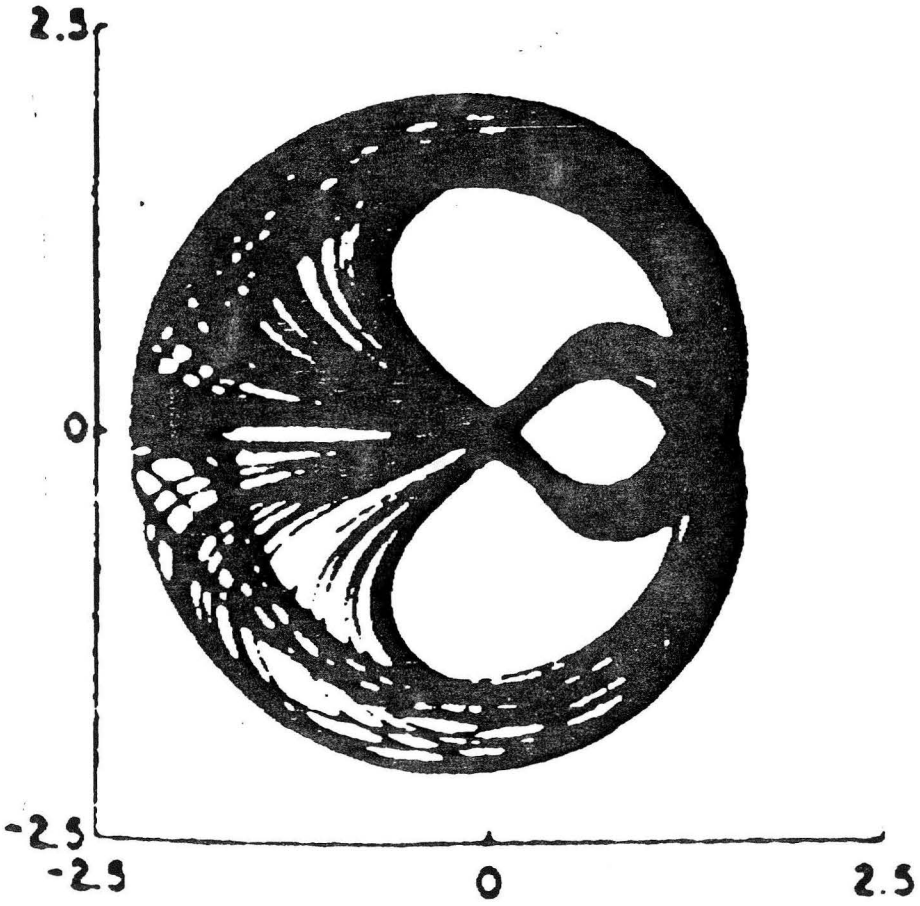


$T = 1.0225\ T_0$

assisted by Otto Friedrich - machinist and carpenter for the laboratory - made this pair of double pendulums in tandem, in accordance with a description given in the American Journal of Physics in 1992. Each double pendulum consists of an upper and lower part, turning on bearings, so that the friction is small. Each part of each double pendulum, both the upper and the lower, has its own natural period for small-angle oscillations. The situation for the lower pendulum is similar to that of the perturbed simple pendulum I previously described. But now the perturber (the upper part) is itself significantly perturbed; we have what is called feedback, circular causation.

The two double pendulums are identical twins. They are mounted together on a sturdy support, so that neither will influence the other. If I start one of them in an oscillation, the other does not pick up the motion. If I start both together in a small oscillation, they play together nicely.

Now let's try a large initial displacement. The pendulums don't stay together. Nor, if we try the experiment over again, does either do exactly what it did the first time. I hope this surprises you. Before trying to account for it, let's see how we might show quantitatively that the repetition isn't exact. We could use a rigidly mounted electromagnet to hold the pendulum in a fixed initial position, to the side. Suppose the switch releasing the pendulum started a stroboscopic flash camera, that took photos every 25th of a second. On each exposure we could measure the angular deviations from the vertical of the upper and

lower parts. Then we could proceed to compare different trials. This has actually been done.

In explaining this, I shall introduce a bit of the relevant mathematics. Take a look, for just a moment, at the differential equations of the double pendulum (Figure 5). On the lefthand side, on top, you see $\ddot{\theta}_1$, the angular acceleration of θ_1, which is the deviation of the upper pendulum from the vertical. And below, on the lefthand side of the second equation, you see $\ddot{\theta}_2$, the angular acceleration of θ_2, the deviation of the lower pendulum from the vertical.

On the righthand sides you see a number of constants: g, l_1 and l_2, the lengths of the two parts of the pendulum; μ, the ratio of their masses. Also involved are sines and cosines of the two angles, and also of their difference, $\Delta\theta$; these are variables. And two more variables are $\dot{\theta}_1$ (over-dot) and $\dot{\theta}_2$ (over-dot), the momentary angular velocities of the upper and lower parts of the pendulum. Both $\ddot{\theta}_1$ (double-dot) and $\ddot{\theta}_2$ (double-dot) thus depend on four variables, θ_1 and θ_2, $\dot{\theta}_1$ (dot) and $\dot{\theta}_2$ (dot).

The two equations are not soluble; we can't get separate formulas for the two angles, as functions of time. But our system at any moment depends on the four variables θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$. In the 1830s William Rowan Hamilton proposed representing the evolution of such a system in a hyperspace, with a number of dimensions equal to the number of variables on which the state of the system depends. Then a point in this space would correspond to a momentary state of the system, and a succession of points,

$$\ddot{\vartheta}_1 = \frac{g(\sin\vartheta_2\cos(\Delta\vartheta) - \mu\sin\vartheta_1) - (l_2\dot{\vartheta}_2^2 + l_1\dot{\vartheta}_1^2\cos(\Delta\vartheta))\sin(\Delta\vartheta)}{l_1(\mu - \cos^2(\Delta\vartheta))},$$

$$\ddot{\vartheta}_2 = \frac{g\mu(\sin\vartheta_1\cos(\Delta\vartheta) - \sin\vartheta_2) + (\mu l_1\dot{\vartheta}_1^2 + l_2\dot{\vartheta}_2^2\cos(\Delta\vartheta))\sin(\Delta\vartheta)}{l_2(\mu - \cos^2(\Delta\vartheta))}.$$

or trajectory, would show how the system develops. The space is called phase space. In our case, the space is 4-dimensional, and you can't visualize it. You can nevertheless conceive it without contradiction. In the 1890s Henri Poincaré undertook to study insoluble, nonlinear differential equations, by examining the ensemble of possible trajectories in phase space.

In our case, let's consider a few trial runs with our double pendulum, say four, and compare the points in phase space at the successive moments when the photographs are taken. For a given moment, the points in the four different trials will not be the same. There will be a "distance" between any two of them, and we can get numbers for these distances, using the 4-dimensional analogue of the Pythagorean theorem; that is, we take the square root of the sum of the squares of the components. In this figure (Figure 6), the experimental separations are plotted for the first half second. The solid line was obtained by numerical integration of the differential equations, using two slightly different sets of initial conditions. The separations increase on the whole. That the separations have downswings at certain places is due to the fact that, at the end of each swing, when the lower pendulum is starting down again, it pulls down on the upper pendulum, and this is a relatively stable situation.

A statistical study of these numbers shows that, on the average, the separation increases geometrically. Suppose the initial separation between two trajectories in phase space is Δx_0. Then the separation at time t is

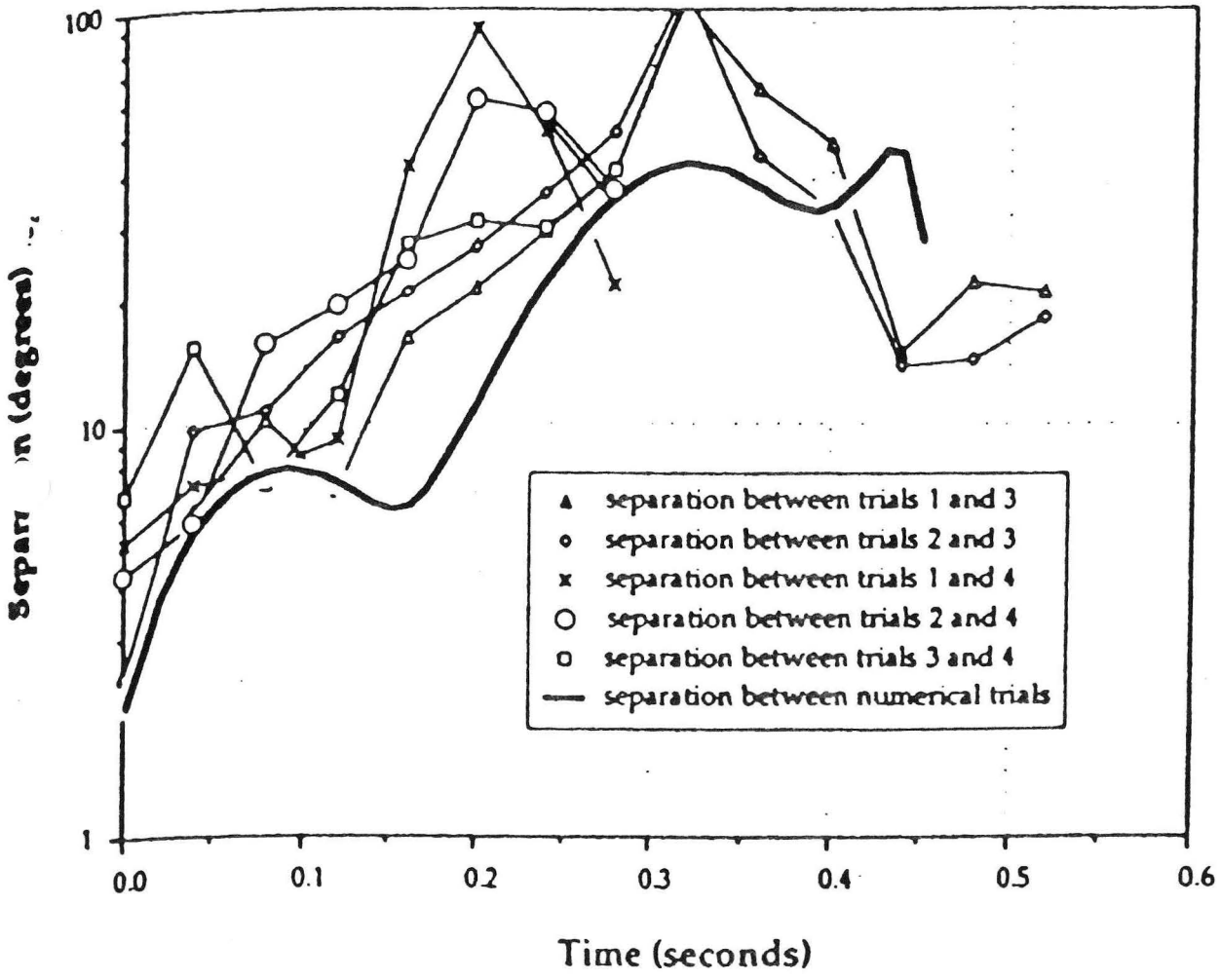


Fig. 9. Experimental results from typical initial condition; bold curve is numerical result.

$$\Delta x_t \approx \Delta x_0 \cdot e^{\lambda t},$$

where e is a constant greater than 1, and λ is a positive constant, called Lyapunov's exponent, after the Russian who first discussed its import. The separation doesn't just increase by the same additive increment in each unit of time, but gets multiplied by the same factor, so that the increase is exponential.

Of course, our data is only for the first half second. Strictly speaking, λ should be determined as a limit as t goes to infinity. But you can't get funding for experiments that long. The chaos is apparent, but is it real? How explain it?

First, however, what does a positive λ do to prediction? Recently it has been shown that the long-term orbital evolutions of the inner planets, Mercury, Venus, Earth, and Mars, are characterized by positive Lyapunov exponents. For instance, certain perturbations of the Earth are in near-resonance with its annual motion, in close analogy with the case of our perturbed simple pendulum; and the same kind of chaos results. Now we never know, with infinite precision, where a planet is. By numerical integration it has been shown that initial uncertainties for the Earth increase by a factor of 3 every 5 million years. An initial error of 15 meters produces an error of 1.5 million kilometers after 100 million years.

Yes, we'll all be dead, but my concern is a theoretical one, about the nature of our knowledge. Can we understand a little better what this chaos is, and whence it comes?

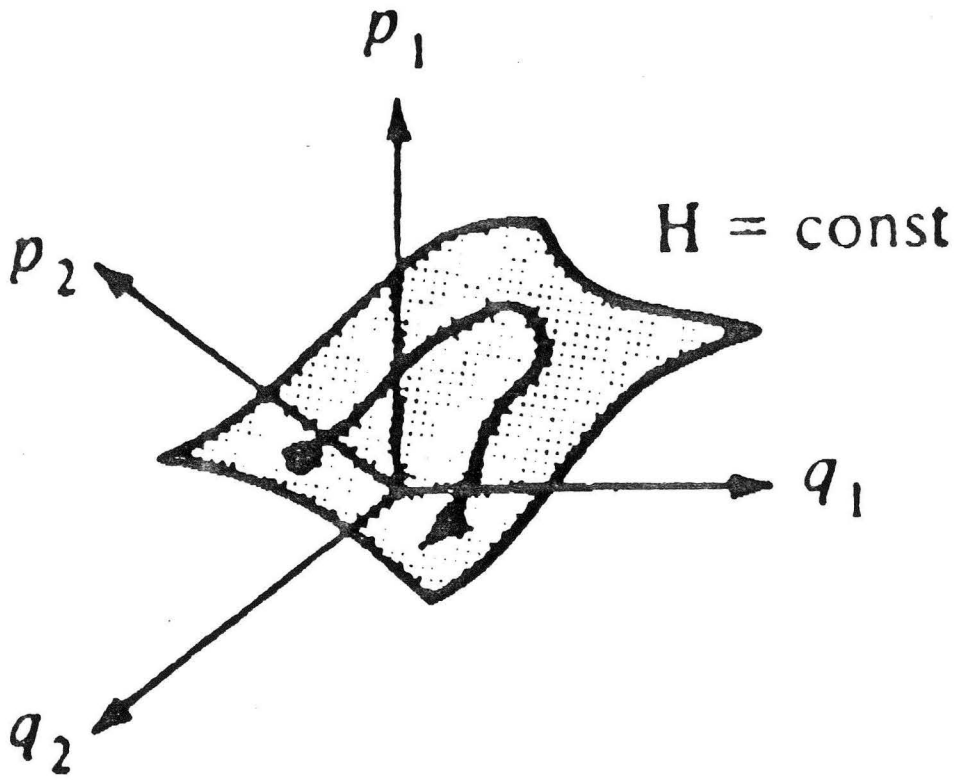
Back to phase space and another of Poincaré's new

techniques. Here (Figure 7) is the 4-dimensional phase space of our double pendulum, somehow represented in a pseudo-diagram; q_1 and q_2 are the coordinates, in our case angles, and p_1 and p_2 are the corresponding momenta, products of velocity and mass. The presentation of the equations of dynamics in terms of the p's and q's is due, once more, to William Rowan Hamilton. H, called the Hamiltonian, is the energy of the system, expressed in terms of the p's and q's. We'll assume for the present argument that H is a constant: $H = H(p_1, p_2, q_1, q_2) = \text{const.}$
In the pseudo-diagram H is represented as a surface; but it is really a hypersurface in 4-dimensional space; the pseudo-visualization is only to help you identify the terms I am using.

The constancy of H will allow us to express p_2 as a function of p_1, q_1, and q_2: $p_2 = p_2(p_1, q_1, q_2)$. We can thus consider the projection of any possible trajectory in the 4-dimensional phase-space onto a 3-dimensional volume. That projection will contain all the information that the 4-dimensional trajectory contained. The reduction in the number of dimensions brings us back to something visualizable.

Poincaré carried the process a step further. If the trajectory is bounded - doesn't go off to infinity - then its projection in the 3-dimensional space will intersect some plane in that space repeatedly, say the plane $q_2 = 0$ (Figure 8). Such a plane is called a Poincaré surface of section. What sort of pattern will the intersections make?

In the 1960s two astronomers, Hénon and Heiles, were


p_1
H = const
p_2
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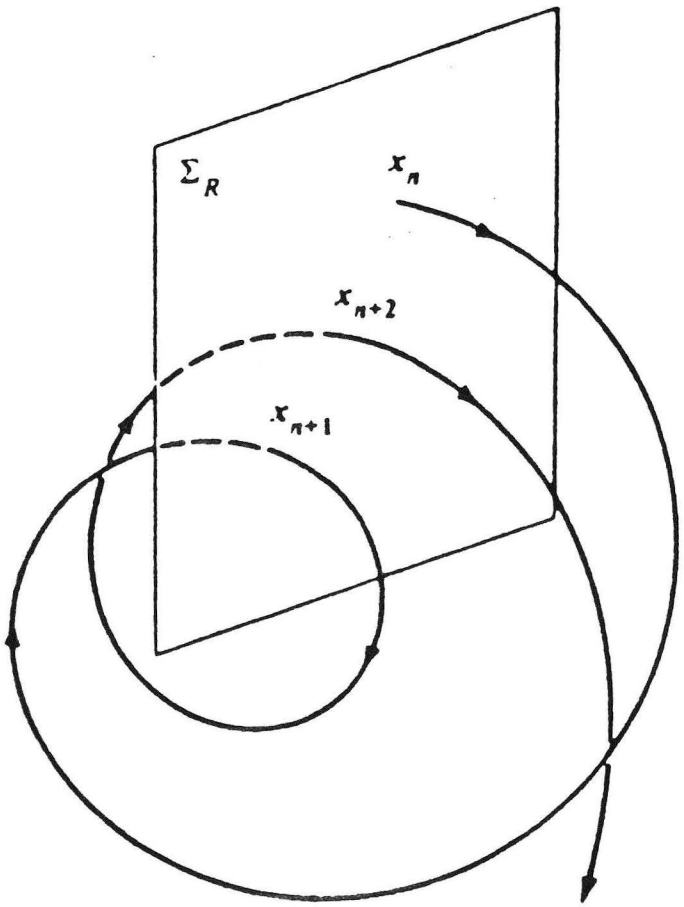
Σ_R
x_n
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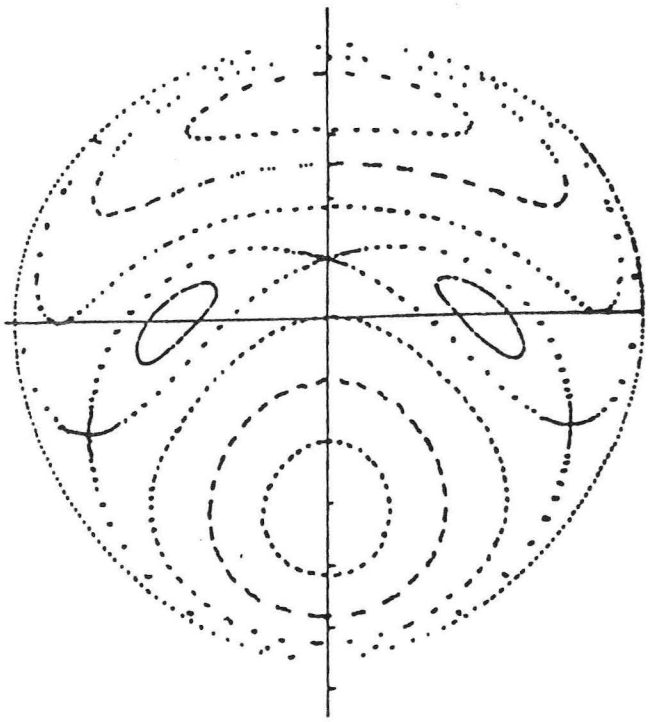
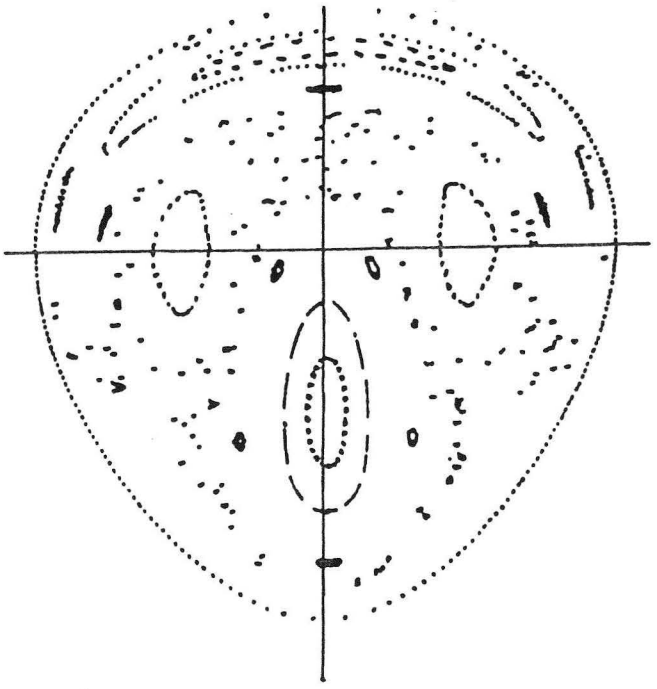
studying a nonlinear differential equation intended to model the motion of a star around a galaxy. They used numerical integration to find successive intersections of the star's trajectory with a Poincaré surface of section. When the energy of the system was relatively small, the intersections lay on certain distinct curves (Figure 9). With an increase in energy, the pattern became this (Figure 10). There were still islands where, for certain initial conditions, the trajectory remained on nice curves; but other trajectories proved to be chaotic, giving seemingly randomly placed intersections. When the energy was increased still further, the islands disappeared (Figure 11).

The motion is deterministic; that is, given the state of the system at any moment, the equation of motion determines its state at the moments that follow. The initial conditions determine a position and a velocity, and the equation of motion then determines an acceleration. Given position, velocity, and acceleration, there is only one way to go. But the resulting pattern looks crazy. Again I ask, How should we understand that?

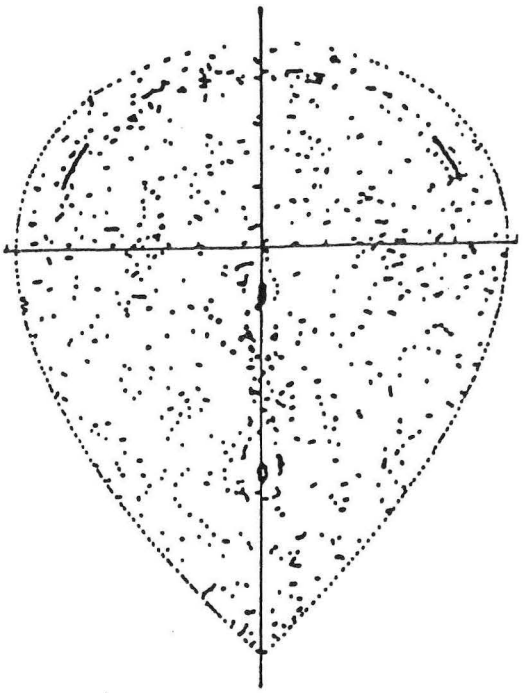
Consider a plane of section with the dimensions p and q; suppose the successive points of section are confined to the unit square (Figure 12). A theorem in Hamiltonian dynamics says that, if the energy of a dynamic system remains constant, the volume occupied by the allowed trajectories is also constant. But in truly chaotic dynamics, the trajectory never returns to the same point, or even to any identifiable curve. The volume of allowable paths gets dispersed, mixed up with bubbles of the unallowable.



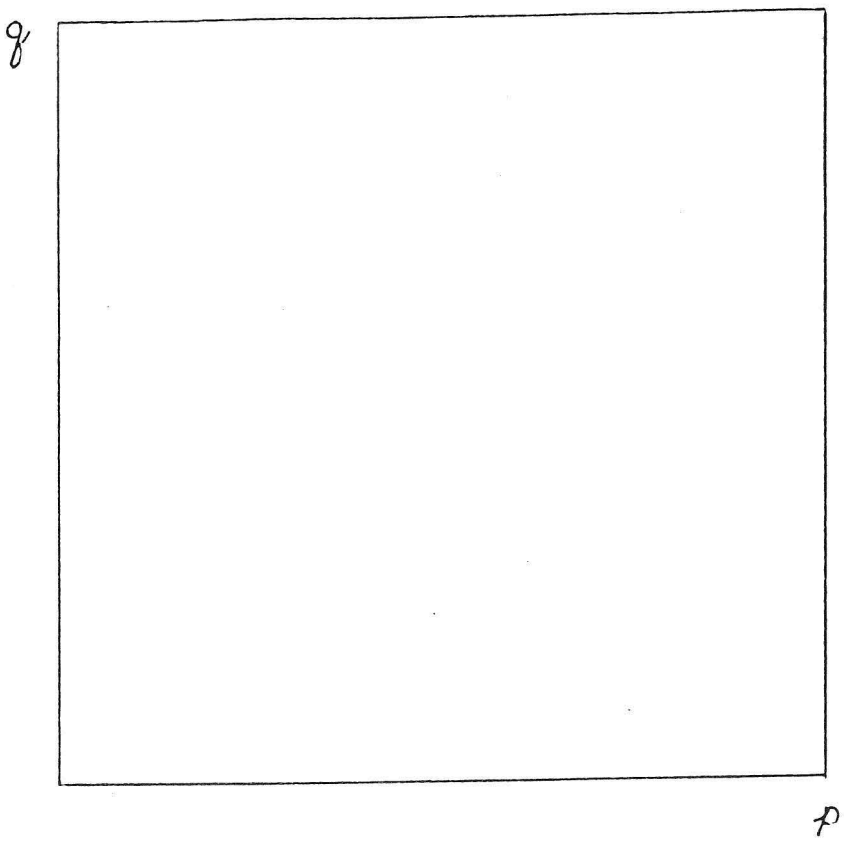
Results for E = .04167



Results for E = .125



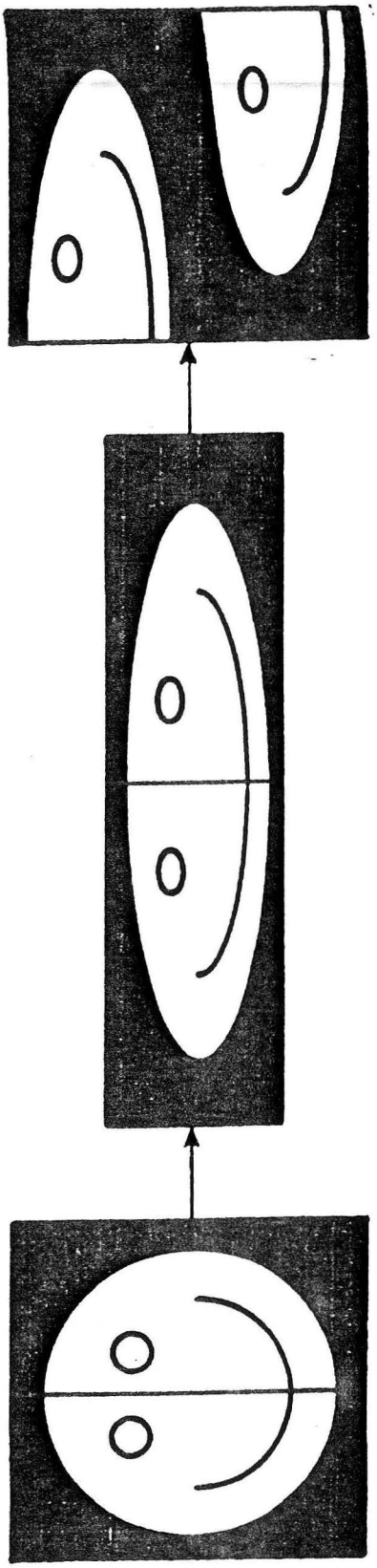
Results for E = .16667


q
p

Some Russian mathematicians have sought to describe this mixing, using a cocktail shaker, rum and cola. Sorry, we're stuck with their noxious example. Initially, the rum and cola are separate. After a few shakes, if we imagine the cola as divided up into moderately small cells, we find some rum in each cell. Later, the subdivision can be made finer and finer, with each cell containing some rum. This is called a mixing transformation. Some dynamical systems have been proved to evolve in this way, for instance a gas consisting of spherical elastic molecules. Nearby possible trajectories necessarily spread apart, defocuss. But, because their energy is finite, they don't go off to infinity, but get folded back into the same space. The rule is Stretch and Fold. It may be characteristic of chaotic dynamics generally. For illustration, I shall present a particular mixing transformation, called the Baker's Transformation (Figure 13). You'll find a description of it on the other side of the handout.

Let the square, as if it were dough, be first squashed down into a rectangle twice as long and half as high; then let the right half of it be set atop the left half. We get back a square. What happens to a point (p,q) within the square? Both p and q are numbers between 0 and 1. By the squashing, all p's are doubled; but then, by the operation of placing the right half atop the left half of the rectangle, the p's that became greater than 1 are reduced again, by subtraction of the unit 1, to numbers between 0 and 1. We can write this: $p \to 2p \pmod 1$.

As for the q's, if our q was in the left half of the



original square, it is simply halved: $q \to q/2$. If it was in the right half, after halving it we add 1/2: $q \to q/2 + 1/2$.

It will be helpful to think about p and q as written in binary notation, so that each p and q will be written as, first, a zero, followed by a binary point (replacing our decimal point), followed in turn by a string of zeros and ones, infinitely long. All numbers between 0 and 1 can be written thus. 0.1 means 1/2, 0.01 means 1/4, 0.001 means 1/8, and so on. We use powers of 2 instead of powers of 10. Let our initial p and q be:

$$p = 0.p_1p_2p_3\ldots, \qquad q = 0.q_1q_2q_3\ldots,$$

where the letters with subscripts are zeros and ones. Now a neat thing about binary notation is that multiplying by 2 just amounts to shifting the binary point to the right, while dividing by 2 just amounts to shifting it to the left. If our initial point was in the left half of the square, then p_1 was 0, and after the squashing, the transformed p will be

$$0.p_2p_3p_4\ldots$$

But the same result holds if our initial point was in the right half of the square, for then p_1 was 1, and after the binary point is moved to the right, 1 must be subtracted. In successive transformations p will become

$$0.p_3p_4p_5\ldots, \qquad 0.p_4p_5p_6\ldots, \qquad \text{and so on.}$$

What about the q's? If our original point was in the left half of the square, then we want the halved value of q, which is

$$0.0q_1q_2q_3\ldots.$$

I have moved the binary point to the left one place. If our

original point was in the right half of the square, then q first gets halved, but we must add 1/2 when the right half of the rectangle is put atop the left half. Now, in this case p_1 was 1, which in the first binary place after the binary point, means 1/2. So the transformed q can be written $0.p_1q_1q_2q_3....$ Actually, this works for q's in the left half of the original square as well, for there p_1 was 0. A little reflection will show you that the succession of transformed q's will be

$0.p_1q_1q_2...$, $0.p_2p_1q_1q_2...$, $0.p_3p_2p_1q_1q_2...$, and so on.

As the successive pairs of transformed p's and q's emerge, the important digits, the digits up front, come from ever farther to the right in the original coordinate p. Initially, they looked insignificant. Yet however far to the right they were originally, they become crucial as the returns continue. The sensitivity to initial conditions is infinite.

There is one more thing I want to say about the p's and q's. You perhaps know that if, after any place in the sequence of digits representing a number between zero and 1, we get a repeating pattern, going on indefinitely, then the number is expressible as a simple fraction. You can write it down exactly, without taking an infinite time to do so. There are other numbers, like $\sqrt{2}/2$, that you can compute to as many places as you like; there is an algorithm, a computational procedure, for doing so. There are even algorithms for computing $\pi/4$, which is neither expressible as a fraction, nor even as the root of an algebraic equation.

All these are known as computable numbers. The algorithms for computing them can all be expressed with a finite number of mathematical symbols. The algorithm may involve some kind of infinite recursion or iteration; but this can be indicated with a single symbol, meaning, "keep doing it over again." So all the algorithms can be written in what we can call an alphabet, with a finite number of symbols; and we can then alphabetize them, list them in an order so that none will be left out, if we keep on going. Thus they are denumerable. Therefore computable numbers form a denumerable set. It is, however, an infinite set; every integer, for instance, has a square root, so there are an infinity of them. But there is a proof, Georg Cantor's famous diagonal proof, showing that the numbers between 0 and 1 are infinitely more numerous than a denumerable set. It follows that there are infinitely more incomputable numbers than computable ones. Our chaotic trajectories have landed us in the midst of this jungle of incomputable numbers.

Suppose, though it is empirically impossible, that we knew our initial p and q exactly. The differential equation for the motion is not soluble; our only resource is numerical integration, for which we turn to a high-powered computer. The computer, however high-powered, cannot give us the chaotic trajectory precisely. That is because it is a finite-state machine. It cannot accept a number expressed by an infinite number of digits; it automatically rounds it off. With our p's and q's undergoing a mixing transformation, the rounding, after a

while, will be disastrous; we will lose essential information. Whether the p or the q, at some later stage in the succession of transformations, starts with a 0 or a 1 will be as uncertain as the toss of a coin.

Let me now, by way of conclusion, state some thoughts as to the import of nonlinear dynamics.

1. According to Laplace, writing in 1812 (and I quote),

> An intelligence that knew, for a given instant, all the forces by which nature is animated, and the respective situation of all the beings that compose it, if it were vast enough to subject these data to analysis, would embrace in a single formula the motions of the largest bodies of the universe and of the smallest atom; nothing would be uncertain to it, and the future, like the past, would be present to its eyes. The human mind, in the perfection that it has been able to achieve in astronomy, presents a pale image of this intelligence.

Laplace is expressing a universal determinism, which he equates with predictibility. Such a determinism, presenting the world as a closed causal network, has often been taken as a dogma of science; but its universalism appears to make the activity of the scientist unintelligible.

> Voltaire swallowed the doctrine whole:
> everything [he wrote] is governed by immutable laws ... everything is prearranged ... everything is a necessary effect.... There are some people who, frightened by this

> truth, allow half of it... There are, they say, events which are necessary and others which are not. It would be strange if a part of what happens had to happen and another part did not.... I necessarily must have the passion to write this, and you must have the passion to condemn me; we are both equally foolish, both toys in the hand of destiny. Your nature is to do ill, mine is to love truth, and to publish it in spite of you.

Post-Newtonians, impressed by the success of the new dynamics, did not consider that the new methods might prove limited in scope.

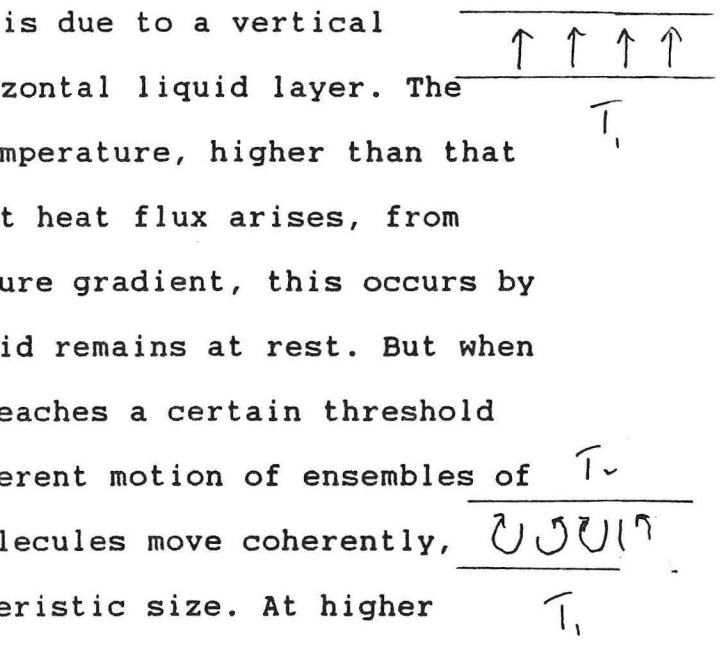
The success of this dynamics was a success in solving linearized differential equations. The world was taken to be an integrable system, each variable being finally expressible as a function of time, independent of the others. So the world would be made up of non-interacting Leibnizian monads, each experiencing its own private cinema, the harmony between them divinely preestablished.

This view, I say, was mistaken, because the differential equations required to model processes in the real world are mostly nonlinear, and most nonlinear differential equations are insoluble. It is from insoluble, nonlinear differential equations that dynamical chaos arises. Here determinism and predictibility part company; Laplace's demon, to do what he required of it, would need to compute with incomputable numbers. Successive approximations, which are the human way, wouldn't suffice.

I have spoken so far as if of a closed dynamic system, insulated from the rest of the universe. But mixing systems such as I have described are hypersensitive to initial conditions, and therefore hypersensitive to tiny perturbations. The flash of an electron in a distant star may affect our mixing system. The intelligence that Laplace imagined, however vast, being yet discursive, will suffer from overload. If there is a God that knows the future, it is by means inscrutable to human reason.

2. I want now to go beyond chaos. There is more to nonlinear science than chaos. As we have seen, the chaos we have been concerned with is not simply disorder; it is approached in an orderly way; it is describable in a coherent way. Can nonlinear dynamics lead to more interesting sorts of order? In fact, in dissipative systems far from equilibrium, new and surprising kinds of order arise. Some of these are described in a book by Prigogine and Stengers entitled Order out of Chaos.

Example. The Bénard instability is due to a vertical temperature gradient set up in a horizontal liquid layer. The lower surface is heated to a given temperature, higher than that of the upper surface. Thus a permanent heat flux arises, from bottom to top, and for a low temperature gradient, this occurs by heat conduction alone, while the liquid remains at rest. But when the imposed gradient of temperature reaches a certain threshold value, a convection involving the coherent motion of ensembles of molecules is produced. Millions of molecules move coherently, forming convection cells of a characteristic size. At higher



temperature gradients there occur periodic fluctuations in temperature and in the spatial arrangement of the cells; finally there is turbulent chaos, which, again, is not without its order.

Similar kinds of order arise in dissipative chemical systems. With reactants entering and products leaving, the system may organize itself spatially, or may come to act like a chemical clock, beating rhythmically. Such coherent behaviors on the macroscopic level do not appear to be reducible to the dynamics of atoms and molecules. We have what may be called emergence.

3. According to a fairly broad consensus among scientists today, living things are among the entities that have so emerged. We are, in some sense of the word "are," stardust. Living things are complex, dissipative structures, to some extent self-regulating, but maintained ultimately by the flux of energy from the Sun. If the geological time-scale is represented as a 30-day month, then life appeared in the oceans by the 4th day, but became abundant only on the 27th. The first land plants and the first vertebrates appeared on the 28th day; most of human culture appeared only in the last 30 seconds. All this, at least up to the last 30 seconds, is understandable in terms of evolution by random variation and differential reproductive success.

Living things are not only embedded in the surrounding geological world, but by their activities they have altered that world; at an early stage, for instance, ancient relatives of present-day algae produced the oxygen of the atmosphere. In various degrees, living things have a circumscribed autonomy; it

is wider for those with homeostasis of the blood, wider still for those that can reason before reacting. These are beings with desires, aims, purposes. The world-lines of such semi-autonomous entities, with their separate agendas, may intersect. A man goes to the agora, in the case imagined by Aristotle, and meets someone who owes him money. Neither planned this encounter; it is by chance. Species migrate or spread, encounter one another, interact, find new ecological niches.

A world evolving through chance-like encounters, in which new entities emerge in time, including intelligent beings, is unintelligible if, with Leibniz or Voltaire or Laplace, we take that world to be an integrable system. In an integrable system, the mere reversal of velocities sends time backwards. There is no essential distinction between future and past. The smoke can go down the chimney and reconstitute the firewood.

Why can it not? In the middle of the 19th century the law of entropy was discovered: in any closed system, a certain mathematical function tends to a maximum, and there is thus a forward direction to time, diametrically opposed to the backward direction. This law is of everyday use in physics and chemistry, to predict the outcome of experiments. But it contradicts dynamics, if the world of dynamics is an integrable system. Physicists like Boltzmann sought to derive the law of entropy from dynamics; irreversibility from reversibility. By the 1890s it was clear that it couldn't be done; a <u>statistical</u> or <u>probabilistic</u> assumption, distinct from the dynamics, was

necessary. Chance had to be assumed to be real. This is an empirical assumption, warranted by experience. Probability theory, at its core, is an empirical science, which assumes the future to be different from the past.

Also toward the end of the 19th century, the American philosopher C.S. Peirce suggested that the dissipative tendencies of entropy could be balanced by the concentrative effects of chance. Chance can have an integrative role, in the emergence of new entities like us. Thus dissipative processes intertwine with integrative ones.

Nowhere is this more the case than in the activity most distinctive of humans, that of learning and communicating. It is not possible without a functioning brain, dependent on a flux of energy; the reactions involved are dissipative and therefore irreversible. When we learn a Greek paradigm, we change the physiology of the brain. We learn not as beings detached and separate from the world, but as parts of it, by engaging in activities of exploration, hypothesis, construction, testing. Here there is an interplay between chance and reason. And this is especially true in learning about nature: such learning requires that we enter into a dialogue with nature. Thus I think that Einstein, when he sought a vision of the world from totally outside it, and denied the reality of time, was mistaken.

The perspective I am suggesting leads to a new respect for nature, of which we are not the overlords but in which we are both embedded and emergent. In this perspective, there are no

guarantees; we live in a chancy world. Nevertheless, a qualified hope is rational. Human knowledge increases, not always steadily, sometimes by surprising zigzags or even reversals; but the trend is unmistakably incremental. It is not a deductive chain. It is a rope, no single strand of which is, by itself, of incorrigible strength; but different strands, by pulling against one another, constitute a fabric stronger than any of its parts.

In this lecture I have sought to signalize a mistake into which dynamicists and philosophers of the past three centuries fell, imagining the world to be an integrable system. The biological perspective I have been sketching can be a corrective. Our survival depends on recognizing and respecting our own complexity and that of the nature in which we are both embedded and emergent.

According to rabbinical commentary, the first word of Genesis, Beréchit, means not "In the beginning,", but "In a beginning." Twenty-six attempts, say the rabbis, preceded the present Genesis; all ended in failure. Holway shéyaanod, exclaimed God as he created the world: "Let's hope that this time it works."