Identicality: An Interpretation of Quantum Theory

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I. INTRODUCTION

The interpretation of quantum theory remains one of the most difficult and fascinating problems raised by the developments of modern physics. To some extent this is because quantum theory, like the theory of relativity, involves a view of the world which departs markedly from that of classical Newtonian physics. But while relativity rather quickly commanded assent, quantum theory involves so great a departure that it remained unsatisfactory to Einstein and Schrödinger, both of whom struggled to find some way around the strangeness of the theory that they themselves had founded.

In the case of relativity there is some concession to the human need to visualize in the space-time diagrams introduced by Minkowski. One can still follow the "career" of a point mass in such a diagram as the world-line traced out by its history in the four-dimensional space-time. But there is no such visualization possible in quantum theory, for it has proved that every such attempt inevitably leads to crucial misunderstandings. Paul Dirac has written that

> The new theories, if one looks apart from their mathematical setting, are built up from physical concepts which cannot be explained in terms of things previously known to the student, which cannot even be explained adequately in words at all.[1]

More than one of the founders of quantum theory has persuaded himself that the best approach is simply to abandon the attempt to render the theory understandable. This is expressed provocatively by Freeman Dyson in his observation that finally "the student says to himself: 'I understand quantum mechanics,' or rather he says: 'I understand now that there isn't anything to be understood.'"[2] Richard Feynman puts the matter more reflectively when he says, after presenting his summary of the basic "rules" of quantum theory:

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> One might still like to ask: 'How does it work? What is the machinery behind the law?' No one has found any machinery behind the law. No one can 'explain' any more than we have just 'explained.' No one will give you any deeper representation of the situation. We have no ideas about a more basic mechanism from which these results can be deduced.[3]

In this lecture I will try to suggest a "deeper representation" of quantum theory. What follows is an interpretation which does not intentionally diverge from the standard results of this theory but which endeavors to approach it from a new direction, and to present it from a new vantage point. After doing this I will comment on the relation of this presentation to the historical development of the theory and to the way it has been previously understood. Finally I will offer some suggestions which encompass the theory of relativity in this new understanding.[4] Throughout the argument will be presented avoiding mathematical formalism so that the problem of the intelligibility of the theory might not be lost from sight. In the process a certain exactitude will be lost, since the mathematics bears to such a non-mathematical account something of the relation of poetry to prose. Perhaps one can demand that poetry be paraphrased in prose, but something important is lost.

II. THE PRINCIPLE OF IDENTICALITY

Ever since the earliest Greek speculations on the atomic theory there has been a question concerning the identity of the atoms. Democritus taught that "atoms differ from each other in shape and size" and Lucretius followed him in that. On the other hand, Dalton asserted that "the ultimate particles of all homogeneous bodies are perfectly alike."[5] This assertion not only has a persuasive and rigorous simplicity about it but also seems warranted by reasoning about the invariant chemical properties of elements. Such reasons moved Newton to write of

> solid, massy, hard, impenetrable, movable Particles ... incomparably harder than any porous Bodies compounded of them; even so very hard, as never to wear or break in pieces; no ordinary Power being able to divide what God himself made one in the first Creation.[6]

Maxwell continued this line of thinking in arguing that each molecule of a given element must share what he calls "perfect identity" with all others of the same element. In his *Theory of Heat* (1872) he argues that this cannot be merely true in a statistical sense of averages in a huge population.

> For if of the molecules of some substance such as hydrogen, some were of slightly greater mass than others, we have the means of producing a separation between molecules of different masses, and in this way we should be able to produce two kinds of hydrogen, one of which would be somewhat denser than the other. As this cannot be done, we must admit that the equality which we assert to exist between the molecules of hydrogen applies to each individual molecule, and not merely to the average of groups of millions of molecules.[7]

He concludes that "the individuals of each species [are] similar to each other, and no intermediate links are found to connect one species with another by a uniform gradation." He makes an interesting comparison to Darwinian evolution, in which such "intermediate links" must be supposed to have existed in order to show how one species might evolve from another. He continues:

> in each [biological] species variations occur, and there is a perpetual generation and destruction of the individuals of which the species consists. Hence it is possible to frame a theory to account for the present state of things by means of generation, variation, and discriminative destruction. In the case of the molecules, however, each individual is permanent; there is no generation or destruction, and no variation, or rather no difference, between the individuals of each species.

It is my suggestion that in the notion of such "perfect identity" of the fundamental elements of nature lies a fundamental principle with profound consequences. Let me then hypothesize the following *principle of identicality*: the fundamental elements of physical reality are classes of identical and indistinguishable states. I apologize for the awkward term "identicality," but it avoids certain misleading connotations of the simpler term "identity." I cannot speak of a "principle of identity" since the point of the utter sameness of the individuals of a certain species is that they *lose* what is usually called their "identity." They are no longer in principle distinguishable each from the others in that species. I will avoid the term "indistinguishable" only because it connotes that it is merely the fault of the *observer* that the individuals cannot be distinguished. By identicality I mean a condition which is not subjective but which inheres in the world even apart from the particular observers, since I suppose that no possible observer, however sharp-eyed, could make the distinctions, that there really are *no* distinctions to be made.

Notice that there are postulated to be not necessarily one but perhaps many "species" or classes of fundamental states. I have used the term "states" to describe the entities which are presumed to be identical because it will shortly appear that a term such as "particle" is inadequate. Sometimes I will call them simply "identicals." If indeed the individuals are identical, any given one cannot be distinguished from any other. But it is precisely such distinguishability which is the central characteristic of what we have usually called particles. For instance, in Newton's Laws it is absolutely necessary that the individual mass points in principle can be labelled and distinguished, for otherwise the Laws cannot be applied without paradox. There is a simple example. Consider a horse pulling a cart. The horse exerts a certain force on the cart. But, according to Newton's Third Law, it would appear that the cart exerts an equal and opposite force on the horse. Then it would seem that the horse could never move forward, since it is subject to two equal and opposite forces. The resolution of this paradox is that the force exerted by the horse must be considered to operate on the *cart*, and likewise the cart's force of reaction operates on the *horse*. By distinguishing carefully on which body the particular forces operate (including the forces of friction) the paradox is resolved, and the horse and cart trundle away. This is only one particular example which is contained in the general statement of the Second Law, that the sum of the forces *exerted on a given body* equals the mass times acceleration *of that body*. In order to give sense to this Law, each body must be capable of being unambiguously distinguished from any other. Ludwig Boltzmann went so far as to call this the "first fundamental principle" of mechanics.[8]

Thus it appears that if the principle of identicality holds rigorously, Newton's Laws cannot apply in its realm. In what way must they be modified? If individuals lose their identity, one might think that, instead of discrete atoms, the world is a continuum of primal matter. But this does not follow. The principle of identicality requires that each class or species of fundamental states comprise states which can be exhaustively identical. Let us consider an experiment such as the Millikan oil-drop experiment, in which evidence is adduced for the existence of a state called *electron* which has a definite charge. Millikan deduced the value of the elementary unity of charge through the careful observation of minute oil-drops charged by static electricity. He himself noted that the value of the charge which he obtained is not a statistical average "but that rather the electrical charges found on ions all have either exactly the same value or else small exact multiples of that value."[9] This is extraordinary since most other measurable quantities show themselves to be subject to statistical variation due to the random error inherent in the act of measurement. Millikan makes his surprising assertion since in thousands of observations the charge on his droplets have only changed by integer multiples of the fundamental charge to an accuracy well within his experimental error. The fundamental principle behind this result is, I suggest, the principle of identicality. If the fundamental charge is always and everywhere identical, it cannot be subject to any variation, for the smallest alteration in it would negate the principle of absolute identicality.

The import of this example is to show that the principle of identicality implies a fundamental discreteness to be found in measureable quantities. This I shall call *quantization*, to give its modern name. It is identicality that underlies the absolute equality of the charge and mass of all the electrons in the world, in our example. Similarly, it underlies the definition of other quantized measures such as spin or strangeness which are necessary to define the fundamental species of states presently known. But as we have seen this quantization co-exists with the denial of the distinguishability that is necessary to formulate particulate laws of force and motion, such as Newton's Laws. So we have retained the discreteness and indivisability which characterized Newtonian particles, but have let go of their distinguishability.

If individual electrons are not distinguishable, this means that each of them cannot be said to have a history or trajectory in the sense that a Newtonian particle travels on a certain definite trajectory through a field of force. Such a trajectory is only possible if the particle that traverses it can always be distinguished from any other particle so that its path can be *continuously* mapped out. If this distinguishability fails, there is no such definite path. This is a momentous consequence that we must explore at length. Imagine a single electron approaching a plate having two very thin parallel slits which are separated by a certain distance. If this electron remains identical to itself during the course of the experiment, there must be one electron found on the far side of the plate some time later. But it cannot be said that the electron passed through only one of the two slits, for that would give it a definite trajectory in a way we have just argued is inconsistent with its indistinguishability. Since we cannot specify which slit the electron went through, it follows that in some way it must have gone through both of them.

If we deny the electron the exact localization of a path, the only other physical quantity that can characterize it would be the magnitude of its momentum (which is always proportional to its energy). For an electron of given momentum, then, there will be a certain distance within which the electron cannot be exactly localized. From considerations of the units one realizes that this distance must be equal to a certain universal constant, whose units are energy times time, divided by the momentum. The constant has the same units as Planck's constant.

Our new conception of the electron is that it is not confined to a definite location (although it always appears undivided at some definite place) and that it traverses distances conveying energy along with it. It is, in effect, a travelling disturbance of finite extent which conveys energy. This is just the definition of a *wave* and our new characterization is that the electron behaves like a wave, in that it travels but is not localized, and that it behaves like a particle, in that it always is observably indivisible. Put another way, classical particles and waves shares between them the essential characteristics of identicality. The classical wave consists of indistinguishable parts, but they are continuous so they lack the aspect of indivisibility. The classical particle is indivisible, but it has a definite continuous path and always thus remains distinguishable from its fellows. So by beginning with identicality we must entertain a new conception of matter which involves aspects of both particles and waves.

One of the characteristics of wave phenomena is that they exhibit interference. This occurs because waves, being identical, can interpenetrate and can then reinforce each other or even cancel each other out. This was long known to occur in water waves. Thomas Young first noticed this phenomenon in the case of light passing through thin slits. He found a characteristic pattern of bright and dark bands produced by the reinforcement or cancellation of the light waves. If indeed electrons have a wave-nature along with a particle-nature then they too should exhibit interference. In so doing they would show that they were not *independent* in the way that different classical particles are independent of each other, in that each classsical particle has *its own* history.

Now the difficulty is that, in the case of water waves, one can see the crests and troughs of the waves, and thus can readily visualize how they can reinforce each other. In one extreme crest mounts on crest to make a super-crest, or, in the other extreme, at some point the waves might negate each other, crest levelled out by trough to make a still point called a node. But where are the crests and troughs of an electron-wave?

In the case of light the crests and troughs are represented by the alternating positive and negative electric and magnetic fields. Even though the fields are invisible, a visual image can be formed at some point in space of an invisible arrow, representing the field, which rapidly flips up and down in swift alternation. Here as with water waves the recognition of a wave is a recognition of a *form*, since a wave is not matter, but a moving form that ripples through matter. In the case of the electron, even in imagination one cannot "see" what could be negative about it, what could represent the trough. This is because the particle aspect of the electron has filled us with a sense of the "positivity" of its being.

A wave must somehow encompass a negative as well as positive being so that the negative part of the wave might interfere with and cancel out the positive part at certain points. By "negative being" I do not mean "un-being." Thinking back to water waves, a trough is the presence in the wave of a *reverse crest*, so to speak. A trough is a disturbance of the water that goes below sea level. So a trough is not a non-being but a being which is reversed compared to a crest. They are mirror images of each other.

Returning to the electron, if we are to give credence to the image of a wave in this case it must be that there is a realm not present to observation in which the adding or cancellation of the waves takes place. Thus behind the externally observable life of electrons we infer that there must be an internal life as well. This internal life is not observable but in it occur the reinforcements and cancellations that result in the observed phenomena of interference. To put it simply, the electron is a wave in private life, a wave which "guides" its public life as a particle.

These appearances are rightly described as the *probability* of finding an electron at some place and time, since certainty about its motion would require following its individual history. This probability is not simply constant, since it is not possible that an electron is equally likely to be everywhere at any time. It is more or less likely to be found at different times and places. One might draw a curve which graphs the greater or less probability of finding an electron in different circumstances. It is here that we find the analogue to the visible shape of a water wave. In the case of an electron it is a *wave of probability*, so to speak.

This probability is external and phenomenological in that it describes appearances as seen by us, the observers. As such, this measure must always be a number which is positive, or at least zero, since our measuring instruments are not capable of registering negative signals. A thermometer reading below zero, for instance, is registering not a truly "negative temperature" but one simply below the conventional zero. In the case of water or light waves the intensity of the wave can be written as the square of the amplitude of the wave. Similarly, in the case of an electron the observed intensity at some point can be written as the square of a *wave function* which describes the wave on an internal level. The wave function gives the inner description of the wave, and its square gives its external, observable manifestation. In the case of water or light waves both the wave function itself as well as its square, the intensity, can be observed by suitable equipment. In the case of electrons, however, the wave function itself is not to be observable, although its square is observable as the probability of detection.

You will notice that I am reneging on my promise to avoid mathematics. This is because the pursuit of the idea of the absolutely identical leads one beyond the world of visible appearance. There is a story that Leibniz, to illustrate his argument against the existence of perfectly identical beings, led the ladies of the court into the woods and challenged them to find two identical leaves. In postulating identicality one is really speaking about things not of this world. We are really speaking about *forms*. The only example of identicality we easily have in common is the integers. The number five, for instance, is always identical, no matter what group of things is being counted as numbering five. But if you will allow me to use numbers I think I can present you with examples that will illustrate what I have been saying about electrons. Just as with electrons, integers are discrete and quantized, are integral and whole (in Latin *integer*). Integers are the observable quantities. Yet underlying them one is drawn to add other magnitudes which are not discrete. I mean here the continuum of magnitudes beginning with those described by Euclid and which span the gap between integers. For Euclid, these are necessitated by the consideration of ratios of integers. Further, one is then led to the introduction of irrational numbers which are not ratios of integers. Other mathematicians continued this development to introduce negative magnitudes, and even quantities representing the square root of negative magnitudes (the so-called imaginary rumbers). One can show that these novel mathematical quantities alone are bound up in a certain way with the integers along with the basic mathematical operations of addition, subtraction, multiplication, and division. Modern mathematicians have argued that these further generalizations of integers constitute a complete and closed collection, sufficient unto themselves and fully general.

Now I might draw the following analogy. The integers that can be counted rest on a basis which is not countable, and which includes negative and imaginary magnitudes. In this modern interpretation the integers are the outer observable outcroppings of an inner network of real and imaginary numbers, positive as well as negative. The integers represent what we can count of a larger, though more inward, mathematical structure. In the same way, the observable states of quantum theory represent what we can observe of an inner, unobservable structure of mathematical forms. There is an internal or noumenal level of reality and an external or phenomenal level. One is knowable, the other is observable. Here I have used Kant's terms noumenal and phenomenal rather playfully, realizing that he means by them something significantly different. The terms as I use them are helpful in suggesting that the inner workings of the world might be intelligible (*noumenon*) in a different way than the outer workings, the phenomena. It can also be shown that the inner behavior of the wave function is governed by a factor oscillating in time proportional to the ratio of the energy of the state to Planck's constant, though the observable energy be constant. In this form Planck's constant shows its significance as the "conversion factor" relating the noumenal to the phenomenal, as the speed of light is a "conversion factor" relating space and time. Through Planck's constant a ceaseless inner oscillation is yoked to outward constancy.

The wave function describing the inner level of the electron is not observable and so is not restricted to be a real number, although its square is positive and observable as the probability of detection. It is possible now to frame equations which will describe the evolution of the wave function such as the Schrödinger equation, which gives a very accurate account of most atomic phenomena, including the structure and bonds of atomic matter. The whole edifice that emerges from the consideration of identicality is called quantum theory.

III. THE HISTORICAL DEVELOPMENT OF QUANTUM THEORY

The perspective I have just outlined has not previously been taken. Previously identicality was considered to be what Feynman called a "beautiful consequence" of other, more basic, axioms. In effect, what I have done is to turn the theory "on its head." I have endeavored to show how one might turn what previously have been considered postulates into consequences of identicality. I do so because it seems to me that identicality is more intelligible than quantization and wave-particle duality. Although the historical development of quantum theory is not a simple unfolding of the idea of identicality, it is interesting to see just what role this idea played.

When Max Planck introduced in 1900 the hypothesis that action must be quantized he was led to do so by an argument stemming from considerations of the significance of irreversible processes in nature. In a series of retrospective lectures he delivered at Columbia University in 1909 he argued that "irreversibility leads of necessity to atomistics," and that the atoms in questions must be "not at all different" from each other.[10] He then extended this argument to radiant heat and light and postulated a parallel "atom" of radiant energy, the famous quantization. It is clear that in his thinking the notion of identicality assumes a fundamental importance. Much about this notion he derived from the preceeding investigations of Gibbs and Boltzmann into the mechanics of large assemblies of atoms.

In fact there is an interesting analogy between the behavior of crowds and the principle we have been discussing. In a crowd the individuals suffer a loss of individuality. As a consequence, the behavior of a crowd exhibits noticeable crests of hysterical enthusiasm and peculiar lulls. In this it is like the interference of the identical electrons. In both cases the behavior of the crowd is quite different than that of distinct individuals.[11] In the case of electrons, however, even one of them is a crowd in that distinct individuality can never be attained.

Although Planck implicitly used the identicality of hypothetical resonators in his argument of 1900 it was not until 1911 that Ladislas Natanson, of Cracow, explicitly noted that of necessity "distinguishability has to be abandoned in order to arrive at Planck's law." Independently, Paul Ehrenfest pointed to the lack of independence of the quanta.[12] The "new quantum mechanics" was set forth in 1926 independently by Heisenberg and Schrödinger. But even before that the notion of identicality had found many important applications. Bose and Einstein had used it in 1924–25 to show how the states of an ideal gas had to differ from the classical predictions, as did Schrödinger himself. In the wake of the new quantum mechanics several crucial implications of identicality came forth. Pauli showed in 1925 how the identicality and attendant exclusion principle for electrons led naturally to the periodic table of elements. By 1927 Heitler and London had also shown that the central mystery of chemical affinity, the covalent bond, depended on the exact identicality of the electrons.[13]

From that time to the present the strangeness of the quantum theory has not diminished even as it has been many times confirmed experimentally. Niels Bohr argued that this strangeness results from the enormous disparity of our macroscopic acts of observation and the otherness of the microscopic realm being observed, a relation he called complementarity. The principle of identicality does not dissolve this strangeness but attempts to find a more intelligible name for it. It rests on empirical evidence in the sense that it is perfectly imaginable that the world might not have been composed of identical states, but experiment seems to show that it is. To all current experimental standards, fundamental quantities such as the elementary unit charge common to all states of matter seem to conform to our hypothesis. But even besides this, the notion of identicality is fascinating because it points to a singular degree of symmetry and unity underlying the observable world, which falls into assymetric and disparate individuals formed of hosts of these identicals. It echoes the form of the Name disclosed to Moses, *I am that I am.*

IV. IDENTICALITY AND THE GENESIS OF SPACE-TIME

Early in our considerations it became clear that identicals do not have a definite path in space and time. This points to a fundamental breakdown of space and time as explanatory elements. Although identicals can be manifest as observable appearances in space and time, I would like to suggest that space and time are not necessary to characterize identicals in themselves. The wave function is best characterized by an abstract manifold, called Hilbert space, rather than by ordinary space and time. This Hilbert space is a "state space" whose axes are the possible states of a system of identicals and in which the observed state of the system is represented by a vector. It is a mathematical image that exists apart from space and time. Moreover, it has in general an infinite number of dimensions, corresponding to the infinity of possible states. From the point of view of identicality it represents the natural representation of the state of the world, from which appearances in space and time are derived. It will prove very significant that the Hilbert space is governed by *Euclidean* geometry. The squared length of vectors in it is the sum of the squares of the components which are observable probabilities and thus are always positive. In contrast, the geometry which Einstein and Minkowski attribute to the union of space and time is that of Lobachevskii. It may be called a hyperbolic geometry in which the length of vectors is only a positive number if the speeds involved are less than that of light.

Let us stipulate that by observation we mean the characterization of a state by strings of integers, such as we can read from instruments. My suggestion is that space and time as Einstein describes them emerge as the states inwardly contained in the Hilbert space are outwardly observed. It seems to me that there is an intermediate term, which is a four-dimensional Euclidean manifold connecting the infinite-dimensional Hilbert space of the states with the four-dimensional space-time of Einstein and Minkowski. Julian Schwinger has shown that if indeed there is such a Euclidean manifold connected to space-time then to every observed species of particle there must correspond a species of antiparticle which has the exactly identical mass, lifetime, and magnitude of all other observable quantities.[14] Essentially, such an antiparticle is a time-reversed state of a particle which can be reached since the Euclidean manfold does not distinguish space from time at all and thus has no causal sense of past or future. This general result on the existence and essential identicality of matter and antimater, called the *TCP* Theorem, has been abundantly confirmed experimentally and is one of the most profound results of relativistic quantum theory.

In this account space-time emerges as the observable shadow of the inner manifold of the identicals. It is particularly significant that Euclid's geometry governs both the beginning point and the crucial intermediate step of the Euclidean four-dimensional space. Euclidean geometry is the inner fundamental state from which the non-Euclidean geometry of Einstein emerges in the course of observation. In this sense the priority of Euclid is vindicated. This result is very much like the purely mathematical observation first noted by Felix Klein that the non-Euclidean geometries of Lobachevskii and Riemann are fully as consistent as Euclid's geometry because they can be shown as transformations, in a certain sense, of Euclidean forms.[15] Euclid's geometry is more intelligible because it is unique, whereas the non-Euclidean geometries all involve a parameter subject to observation. So too in our account Euclid rules the inner realm of the identicals while the non-Euclidean forms emerge in the course of external observation.

This abstract deduction has other important physical consequences. In the Euclidean four-dimensional space there is no invariant distinction between space and time, nor does causality hold. Thus in that intermediate realm between outer appearance and inner determination there can occur *virtual* processes, which are not observable but which can indirectly affect observation. Such virtual processes have been amply confirmed by experiment and considered by theory since the Second World War. For instance, the electron in a hydrogen atom seems to inwardly quiver in response to the virtual fluctuations of charge in the "vacuum." Their condition is like that of Orestes who, speaking of the Furies, says "You can not see them, but I see them."[16] This quivering, called *zitterbewegung*, causes a measurable shift in the spectral lines of hydrogen, called the Lamb shift. Along with the observation of the properties of antimatter, these results bear witness to the presence of a Euclidean manifold underlying appearances in space and time.

V. CONCLUSION

This account attempts to present space and time as not ultimate realities and also to give some sense of the deeper ground from which they proceed. In this account relativity emerges from quantum theory, and both are related to identicality. Up to the present these have seemed distinct and separate strands of physical theory. The new approach also corrects and confirms much of what came before. Einstein objected to quantum theory on the grounds that by denying classical determinism it asserted that "there is no such thing as a complete description of the individual system" and so was incomplete.[17] His objections have a different sense if one considers that, by insisting on determinism, he was still holding on to the essential distinguishability of each particle. In elaborating Einstein's conception of space-time Minkowski spoke of "the everlasting career of the substantial point." This must be abandoned if the points are identical and that career can not be followed. In particular, there can not be "hidden variables" that would undermine the identicality of particles by distinguishing them in some novel way. Bohr pointed out, in reply to Einstein, that the impossibility of "a closer analysis of the reactions between the particle and the measuring instrument" has to do with "a feature of the *individuality* completely foreign to classical physics."[18]

On the other side, we are moved to qualify Bohr's dictum that only what can be observed is real. The principle of identicality rules both the observable phenomena as well as the inner manifold of identical states. Its explanatory power rests in its universal sway.

I do not know whether these considerations will ultimately be sustained by further reflection, or whether they will pass the harder test of showing themselves fruitful of yielding new insights and observable predictions. It seems very important that the fundamental constants of physics, like the speed of light and Planck's constant, have just the value that they do. But no one has any clue to *why* they have these values. Should not a deeper theory tell us what these constants must be, on grounds of inner necessity? Such a theory should also address the great unsolved problem of the reconciliation of the quantum theory and Einstein's geometrization of gravity into curved space-time, both theories so compelling and so different.

In the end, I am not sure that I have helped to make these modern insights more understandable. But it does seem to me that there is something of great significance in the notion of identicality. It calls us to reflect on the deep form of connection, in which may lie the dark backward and abysm, the womb of space and time.

NOTES

1. P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, Oxford, 1958 [fourth edition]), p. vii.
2. F. Dyson, Sci. Am. **199**, 46 (1958).
3. R. P. Feynman *et al.*, *The Feynman Lectures in Physics* (Addision-Wesley, Reading, MA, 1965), vol. 3, p. 1-10.
4. The argument of this lecture can be found in a more complete and mathematical form in a series of papers "The Principle of Identicality and the Foundations of Quantum Theory." Parts I and II deal with the historical developement of identicality in the works of Gibbs

and Planck; Part III gives the argument sketched in this lecture concerning the implication of the general form of quantum theory from identicality; Part IV concerns the relation of the Lorentz group to the underlying Euclidean manifold.
5. See the valuble historical discussion in M. Jammer, *The Conceptual Development of Quantum Mechanics* (Mc-Graw Hill, New York, 1966), p. 338 ff.
6. I. Newton, *Opticks* (Dover, New York, 1979), p. 400.
7. J. C. Maxwell, *The Theory of Heat* (D. Appleton, New York, 1872), pp. 309–312.
8. L. Boltzmann, *Theoretical Physics and Philosophical Problems* (D. Reidel, Dordrecht, 1974), pp. 228–231.
9. R. Millikan, *The Electron* (University of Chicago Press, Chicago, 1963).
10. M. Planck, *Eight Lectures on Theoretical Physics* (Columbia University Press, New York, 1915), p. 52 ff.
11. See E. Canetti, *Crowds and Power* (Farrar Straus Giroux, New York, 1984), pp. 17–22, 29–30 *et passim.*
12. See A. Pais, *Inward Bound* (Oxford University Press, Oxford, 1986), p. 283 and M. Klein, Proc. Amsterdam Acad. **62**, 41 (1959).
13. For further historical details see M. Jammer, Ref. 5, pp. 342–345.
14. See J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, MA, 1970), vol. 1, pp. 42–50.
15. See R. Bonola, *Non-Euclidean Geometry* (Dover, New York, 1955), pp. 177–180, 238–264.
16. Aeschylus, *Choephoroi*, l. 1061.
17. Einstein's "Reply to Criticisms" in P. A. Schilpp, *Albert Einstein: Philosopher-Scientist* (Harper & Row, New York, 1959), p. 671.
18. N. Bohr, Phys. Rev. **48**, 696 (1935); emphasis his.