

A Meditation on Multiplication

It is my honor to give the Christopher B. Nelson Lecture tonight. Mr. Nelson is an alum of the college and was president of St. John's College in Annapolis, beginning in 1991. His presidency was marked by consistent care for our community of learning and lasted for more than 25 years, a remarkably long tenure for a college president.

I'd like to take this opportunity to invite you all to the question period after lecture, to be held in the Conversation Room. We call it the question period rather than the question-and-answer period for a reason, so adjust your expectations accordingly. By tradition, students are allowed to ask the first few questions.

There have been two times in my life that I remember being genuinely confused about what multiplication is. The first time, I was probably about six years old, excited by the prospect of learning the thing called a "times table." What is this mysterious thing that puts lists of numbers together with x's in between, equaling bigger numbers? I remember asking my mother what multiplication was. She said something like, "It means to take something a certain number of times, so that $2 \times 3 = 6$ " means taking two three times. Two, four, six!" Her clear response was very satisfying to me and paved the way for much success in my young career.

As I went through my elementary and then junior high school days, various elaborations on the main idea of multiplication came up. I learned to multiply fractions and decimals, to find areas by multiplying lengths and widths, and take powers of numbers: first squares, such as three squared equals nine, then cubes, then higher powers. By the time I reached high school, the applications of the simple tool of multiplication had, well, multiplied! Without my noticing it, I had completely left the realm of taking something "a certain number of times." Throughout my college and graduate career in the sciences, multiplying in one way or another was second nature. Then, I came to St. John's... which brings me to the second time in my life that I remember being genuinely confused about multiplication.

Our freshmen have already begun their study of Euclid, which will continue for most of the freshman year. Euclidean geometry suffuses the program, coming up in Plato, Kant, Apollonius, Descartes, Galileo, Newton, and Lobachevski, among others. My own time with Euclid began in my first year as a tutor, with a freshman math class. After wrestling with triangles, rectangles, circles, and ratios,

we arrived at Book VII, the first book focusing on numbers. “Ah,” I thought, “What simplicity! What clarity, after the messiness of ‘same ratio’...” Part of the clarity came from Definition 15:

“A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.” (Euclid, p. 157)

This brilliance of clarity evoked memories of my six-year-old self, just beginning to encounter multiplication. My mom’s explanation to me was vindicated by Euclid himself.

Alas, my simple pleasure in the clarity of this definition was short lived. I began to think over the previous books of Euclid. My class and I had been comparing the sizes of triangles, rectangles, and parallelograms for weeks. I suddenly realized that I didn’t really understand how we had been comparing these areas.

In my own mind, with years of experience multiplying length by width to find the area of a shape, I had been tacitly assuming that something like this multiplication was at the root of all our comparisons of areas. I had noticed, of course, that Euclid was not using numerical comparisons, but rather was proving areas to be equal in principle. But the implications of how he was treating areas didn’t sink in until I saw Definition 15. Why did he only need to define multiplication when he wanted to think about numbers? Why hadn’t Euclid defined multiplication earlier? Wouldn’t it make sense that the way we calculate areas would be discussed at some point, while he is working so diligently with areas of various sorts?

Filled with puzzlement about multiplication once again, I realized that Euclid had never calculated the area of anything, that my modern-educated mind had read into Euclid something he hadn’t done: namely, finding extent of an area by multiplying length by width. So I attempted to take that next step myself. How would I use Euclid’s Definition 15 of Book VII to calculate the area of a rectangle, for instance? Can I multiply length by width using definition 15? My lecture tonight will begin with this question.

Once again, Euclid defines *multiplication* thus: “A number is said to multiply a number when that which is multiplied is added to itself as many times as

there are units in the other, and thus some number is produced.” Can I treat length and width in the same way as I treat numbers? If I can, I would have to somehow add length to itself as many times as there are units in the width, to produce the measure of an area. Can I use this process to find an area?

There are two problems that I can see with using the language of “adding to itself” when multiplying length and width. First, there is the problem of dimension. If I’m thinking about this process as adding the same length to itself a number of times, thus producing a longer length, why do I have any reason to think that I would somehow get a measure of area? In other words, how will I relate the measure of the longer length, one dimension, into the measure of an area, two dimensions?

Second, there is the problem of finding a unit in a given length, which, unlike a number, doesn’t come with a given number of units. How do I know how many times to add length to itself if my width isn’t labelled with a unit measure? I’ll address these questions each in turn. The first problem we confront is the problem of dimension.

Let’s start with a simple example. I have a rectangle with length that is three times longer and width two times longer than some length. I will take this third length, that is, half of the width, to be my unit. We’ll leave the problem of how I can find a unit for my length and width for now and just assume we can do it. Following Book VII, Definition 15, I multiply the length by the width. I take the length, which is three units, and add it twice, that is, the number of units that are in the width, ending with a new length that is six units long. On what basis do I think that this long breadthless length would have anything to do with finding a measure of the area of a rectangle formed from my two starting lines? How can I turn one dimension (length) into two (area)?

Let’s look at the problem more closely. I have a rectangle of length three and width two (Figure 1), and I’d like to find a measure of the area. I can draw vertical parallel lines at right angles upon the rectangle at each unit length as well as one horizontal line at the unit width, dividing up the width (Figure 2). I can say now, using Euclid Book II, Proposition 1, which states that the rectangle contained by two straight lines is equal to the rectangles that would be formed if one of the straight lines is divided, that the area of my original whole rectangle is equal to the areas of all of the smaller rectangles added together. (Euclid, p. 37)

At this point, I can refer to the square area based on the unit length as “an area of one” (Figure 3). Since the lines encompassing my smaller areas are all equal, I know that these smaller areas are all equal to each other, that is, they are each an area of one (Figure 4). When I add up all the unit areas, I find that I have six unit-areas, or a final area of six (Figure 5). By restacking my boxes, I can go one step further and represent this area by a rectangle that has a width of our unit and a length of six units, thus bringing the comparison to the line that is six units long into focus (Figure 6).

It doesn't seem coincidental that the same answer, six, came up both when I multiplied length by width and got a longer line of six units, and when I added up the smaller unit areas in the rectangle. In both cases, I'm taking something of three units twice. In the first case, I'm taking a length of three twice, to get a length of six. In the second case, I'm taking a set of three unit-areas twice, to get an area of six. There is nothing in my procedure that relies on the particular numbers I choose. If my length had been 29 units and my width four units, the process would have been the same, just with more divisions. In other words, I seem to have found a procedure showing that the number I get when I multiply the number of units in my length by the number of units in my width will be the same as the measure I get for the area of my rectangle. I haven't produced two dimensions out of one but I have found a reliable connection between the one-dimensional measure of multiplied length and the area that will come from those two lines, if they are set up geometrically as the length and width of a rectangle.

The procedure depends only on finding a unit that measures both the length and the width. My assumption at the beginning was that the width and the length were both multiples of some other length, which I used as my unit.

But what if my rectangle doesn't have this feature? What if there are no built-in units for me to work with? Let's turn now to the problem of finding a unit. First, we'll spend some time with Euclid's definition in the Greek, to be sure that we understand what he means by a unit (Figure 7).

In Book VII, definition 1, we learn that,

μονάς ἐστίν, καθ' ἣν ἕκαστον τῶν ὄντων ἐν λέγεται. (Euclid, vol. 2, p. 279)
(Figure 8)

μονάς ἐστίν: a unit is... The Liddell and Scott lexicon entry for μονάς, as in unit, lists it as the feminine substantive of the adjective “μονος,” meaning solitary, alone. Implied in the word for unit is the idea that it is “one who stands alone.” (Figure 9)

μονάς ἐστίν, καθ’ ἣν: a unit is that according to which... καθ’ is a modification of κατά, because of the rough vowel which comes next. κατά is a preposition, and, as is true for most prepositions, can have many meanings depending on the context. In this case, it is followed by an accusative relative pronoun, ἣν, meaning “which” in English. A good translation of this could be “according to which” or “as to which”. (Figure 10)

μονάς ἐστίν, καθ’ ἣν ἕκαστον τῶν ὄντων: a unit is that according to which each of the beings... ἕκαστον means “each” or “every”. τῶν ὄντων is a participle in the genitive case, so “τῶν ὄντων” is the object of “ἕκαστον.” The participial nature of it is usually rendered in English with the ‘ing’ suffix. The root “ὄντ-” in the Greek is the participial stem of the verb “to be.” It appears to be a substantified word, something like, “the beings” or “the things that exist”.

What sort of beings does Euclid mean? Should I take “τῶν ὄντων” to refer to mathematical objects, to numbers, or to something more broadly interpreted? Taken on the surface, “τῶν ὄντων” seems to refer to anything that is, anything that could be called “one.” Jacob Klein, a former dean of St. John’s College and a philosopher of math, spent a good bit of time investigating the question of what sort of thing is addressed by “τῶν ὄντων”. His conclusion after much study was that Euclid really meant what he said: each of the beings, each of the things that are, anything that can be called “one,” is included in the definition of unit.

The scope of our definition just became much broader than anticipated. While I might have been justified in thinking that my work involved only mathematical objects up to now, the introduction of number has brought in the wider world, including each of the things that exist. Each of them? Really? This seems to be at least one surprising result of our examination of this definition. I suppose it makes sense. If I can apply number to anything, that is, if I can count anything, then anything ought to be within the scope of the work. (Figure 11)

μονάς ἐστίν, καθ’ ἣν ἕκαστον τῶν ὄντων ἐν λέγεται: a unit is that according to which each of the beings is called one. This is the natural translation of the straightforward ending, “ἐν λέγεται,” or “is called one”.

If we have a unit, we must have a “being,” one of “τὸν ὄντων” which we are calling “one” according to the unit. If I consider a being in the world, the fact that I can refer to it as “a being,” or “one thing,” is not taken for granted in this definition. Rather, that fact of my ability to call it one is associated with the unit in question. If I have a triangle, the unit is a triangle, that according to which I call it one triangle. Likewise, if I have a sheep, a hamburger, or a sheet of paper. In those examples, the units are sheep, hamburger, and sheet of paper, respectively. (Figure 12 – blank)

The fact that there is a unit for these things shows that each of them possesses integrity. Are there some things for which a unit is not available? Things which don’t have the integrity of unity, which cannot be called one? How about water, air, sand, gravel? A lake can be counted as one, but is there any such thing as “one water”? We can have one grain of sand, but can we have “one sand”? Substances that flow are very hard to count. We tend to make due with an invented unit: one cup of water, or a 40-pound bag of sand. In these cases, the beings in question just don’t fall into units easily. Therefore, I need to find another, often arbitrary, way according to which I can refer to them as one. In these cases, I’m using the integrity of the cup or the bag as my unit.

In Definition VII.2, we find that,

“A number is a multitude composed of units.” (Euclid, vol.2, p.277)

I may have three sheep, twenty sheets of paper, or six cups of water. We count beings with numbers, which are made up of units. Each thing counted, then, must have some sort of integrity, something by virtue of which we can name it and count it, either natural or arbitrary.

What would that unit be, in the case of a length or width? Does a length have a natural unit, like sheep, or would I need to find something arbitrary, like my measure of a cup of water? Recall that, if I’m going to multiply length by width in order to calculate an area, I need to add length to itself as many times as there are units in my width. Is there anything ready to hand by which to count it? It doesn’t seem so, since there are no natural divisions in breadthless length.

Perhaps I can invent a unit, find something like my cup of water. I’ll just cut my width in half and call the half of my width my unit. Does the half of the width have integrity – is it a being that can be called one? Maybe, but it sounds doubtful.

For one thing, what stops me from dividing the width into four, or seventeen, or one hundred? Even assuming I can invent a unit for my width, which one should I pick? Depending on what my unit is, whether a half or a hundredth of my width, that unit will yield the number of times I'll be adding my length to itself to produce an area. Which unit I choose, then, is critical in determining what area I'll find.

It seems that I'll need to do something more than just invent a unit. Perhaps I *can* discover a unit, find some natural division, some reason for calling this part one of the beings “ $\tau\acute{\omega}\nu \acute{\omicron}\nu\tau\omega\nu$.” If I had a good rationale for picking a unit of length, I could find how many of these natural units are in the width and then take the length that many times to find my area.

How will I discover a being lurking in breadthless length? If I can compare my two lengths, that could help. If my width can be divided into a certain number of units into which my length can also be divided, then I might have some confidence that I've found the correct unit. Let's say that I divide my width into 100 parts and take one of those parts and see if I can divide my length into an even number of those parts. What if that piece doesn't go into the length evenly? Maybe then I try reapportioning the width into 200 parts, or 1,000. Can I be confident that I will eventually find a part that will also divide my length into an even number of parts?

The answer, as we find in Book X of Euclid, is no. Definition 1 of that book defines commensurable magnitudes, as those which are measured by the same measure, and incommensurable magnitudes as those which cannot have any common measure. By “magnitude” he just means something having size, so a magnitude could be a length or an area, or some other mathematical entity with a determinate size. Euclid proceeds, in that very long book, to find and categorize several different kinds of incommensurable magnitudes. It turns out that even if I could put my suspicion to rest, the suspicion that a section of a width will never have the integrity of a unit, I now must admit that I may never be able to find a common measure between my length and my width. Our search for a unit by which we could measure the length and width of some area has failed. The process we outlined before, which depended on the length and width having a unit in common, will not work for every case.

One commonly invoked example of a failure to find a common measure for length is the side of a square and its diagonal. (Figure 13) In modern parlance: if

the side of the square has a length of one, the diagonal is “irrational” with respect to the side, and has a length delineated as the square root of two ($\sqrt{2}$). There is no common measure between one and the square root of two, and no way to count out the square root of two. To visualize this, imagine I had a geometrically exact ruler of length two. (Figure 14) If I measure the side of my square with this ruler, it would come to exactly halfway along the ruler, for a length of one. If I measure the diagonal with this ruler, (Figure 15) it would be quite far past the length of one. To find out exactly how much past one it would go, (Figure 16) I could subdivide my ruler into tenths. I’d find that the diagonal would come to slightly over one and four tenths of the length of the ruler. To find out just how much over one and four tenths the diagonal is, I could make even smaller subdivisions of my ruler. I would find that the diagonal came to just over one and forty-one hundredths of the ruler. To find out just how much over one and forty-one hundredths the diagonal is, I could make *even* smaller subdivisions of my ruler. I would find that the diagonal came to just over one and four hundred fourteen thousandths of the ruler. This process would never end, if I’m trying for mathematical exactitude. If I’m not working with a mathematical object, but with a physical square and diagonal, this process would come to an end whenever I decided to end it, at whatever level of precision either my ruler or my patience would call for. (Figure 17 – blank)

The point is, I didn’t have to go very far to find an example of two lengths which do not have a common measure. So, if I’m looking to find the area of a rectangle made by two given lengths, and I’d like to multiply the lengths, how will I know whether there is a common measure between them or not? How will I know whether I have something like my original example-- in which half of the width goes into the length evenly -- or something like the side and diagonal of a square? My hope of being able arbitrarily to make a unit that would be usable for both my length and my width has been dashed.

Given these difficulties, one might be forgiven for wondering why anyone would believe that multiplication of length by width would even be possible. And yet, we do it all the time. How *is* it possible?

One answer has to do with the realm in which we are working and the precision we require. As I mentioned in the example of the side and diagonal of a square, I may be working in the physical world, perhaps just trying to figure out how big my plot of land is, or my tablecloth. In these sorts of cases, it doesn’t

matter so much whether I can find a unit or not. If I have a table that is three feet long and two feet wide, I can happily use my unit, the arbitrarily chosen “foot.” What if the length and width don’t come out to an exact foot measure? Then I come up, arbitrarily, with inches and smaller subdivisions of inches. If my concern is to figure out if my tablecloth will cover my table completely, I really don’t need to measure the length and width of my tablecloth with much precision. If the length is three feet five inches plus a little more than an inch, I can just include that slop-over in my estimations.

The key word here is “estimation.” In the physical world, every measurement is an estimation, by its nature. I hold a ruler or tape measure up against the table. Implicitly, I know that the edge of the table is not precisely given, nor is the position I place the ruler on going to be completely determined. Baked into my measurement is the sense that there is always some amount of sloppiness. The job is to minimize the sloppiness to the point where it doesn’t matter for the job at hand. With strictly practical problems, I can use my procedure without worrying about whether my unit is natural or whether it goes into my length and width an exact number of times.

If I’m working with mathematical objects, though, the story is different. Mathematics promises to be absolutely precise: if I say the length is two feet in a math problem, it will be two feet, not two feet and a smidge. When I am committed to finding what can be proven to be true, not just what gets the job done in the world of tablecloths and land, I need to be sure that my units are real units and that there is no extra bit.

What has happened in between Euclid and our own time that leads us to think we can multiply lengths? A full picture of the development of mathematical thought during those hundreds of years is too much to handle in one lecture. Mr. Lasell will be speaking in the spring on some aspects of it, so look out for that lecture. For my purposes here, I’ll jump to the 17th century and René Descartes. Our sophomores will read his *Géométrie* in the spring semester, in which he spells out a method that he says is “la même que la Multiplication,” (Descartes, p. 3) that is, “the same as multiplication”. Let’s take a look at what he does.

He begins *La Géométrie* with the broad claim that “any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.” (Descartes, p. 2) He goes on to say

that solving arithmetic problems--that is, problems involving numbers--involves only a few operations, namely addition, subtraction, multiplication, and division, broadly understood. He then makes the analogy to geometrical entities, such as lines and shapes, which I'll call, generically, magnitudes. The implication is that as solving numerical problems involves only these four operations, solving problems of magnitudes will also use some few analogous operations. Adding and subtracting geometrical magnitudes are done in a similar way to adding and subtracting numbers; those operations don't present serious difficulties. Adding a line to a line makes a longer line in a similar way that adding a number to a number makes a bigger number.

Descartes realized, as we have, that multiplication is different (and similarly for division, which I won't be discussing). Rather than try to fit geometrical multiplication into something like numerical multiplication, taking one line the number of times as there are units in the other line, he goes a different way. For simplicity, I'll call his procedure "geo-multiplication."

He presents this geo-multiplication in an offhand way, as if the connection to numerical multiplication is clear. Here is his description of what he says is "the same as multiplication":

"...taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity..." (Descartes, p. 2)

He elaborates further with an example, which we'll take step by step. In the example, he is proposing is to find the multiplication product of two lines, BC and DB. (Figure 18) In the spirit of this lecture, we can think of this operation as finding the area of a rectangle with the given length and width. The naming of these two lines by including the letter B in each one implies that the two lines are touching. Sure enough, in his drawing, Descartes puts the two lines together. (Figure 19) He begins by dividing off a portion of one of the lines to be multiplied (DB) with the point A, and calling the line BA "unity," analogously to "one" (Figure 20). He gives no procedure for where to mark off our "A" here, and, in fact, declares that it can be taken arbitrarily, at any point on the line BD. We'll get back to this point later, but for now, let's just put our A here and keep going.

He joins A and C, (Figure 21) and draws ED parallel to CA (Figure 22). Because E occurs past C in this diagram, we'll just extend the line BC until it hits E (Figure 23). Now, we've done it, according to Descartes. We've geo-multiplied two lines, BC and BD. The product turns out to be the line BE. To understand this, we will need to do a little more work with Euclid.

If you've studied Euclid already, you may remember that one of the properties of similar triangles is that their sides are proportional. In this diagram, the triangle BAC is inside the larger triangle BDE, with the bases being parallel. Because the angles of the triangles are equal, we know that those two triangles are similar to each other (Figure 24). Thus, we know that the sides are proportional, or DB is to BA as EB is to BC (Figure 25) That is, if DB is three times as long as BA, we know that EB will also be three times as long as BC. Once we have proportional sides, we know, by Euclid VI.16, that the rectangle formed by the lines at the outside of the proportion, DB and BC, is equal in area to the rectangle formed by the lines at the inside of the proportion, BA and EB (Figure 26).

The rectangle on the left is formed from the two lines I had wanted to geo-multiply: DB and BC. The rectangle across the top of the figure is formed from the lines EB and BA, or BE and the line which we previously defined as unity, or one (Figure 27). Descartes has found a way to build a rectangle with a width of one and the length of which gives me the area of the first rectangle. This is essentially what we did before, with our restacking of the rectangle that was six units in area. (Figure 28, then 29) In this way, using an arbitrary unit, the measure of the line BE can be thought of as a representation of the size of the area of the rectangle BD, BC.

There is a further comparison between numbers and magnitudes that lies underneath the procedure he uses (Figure 30). In Book VII, Proposition 19, Euclid proves that for any four proportional numbers, the multiplication of the extremes produces a number which is equal to the multiplication of the means. (Euclid, vol.2, p. 318) This proposition is part of a series of propositions in which Euclid proves analogous things about numbers that he proved in Book VI for geometrical magnitudes including length. Book VII.19 is analogous to Book VI.16, in which he proves that the rectangles formed by the extremes are equal to the rectangles formed by the means. In using the rectangles formed by the means and extremes to come to a rectangle whose area can be represented by a length, he is using a process that can also be used with numbers and multiplication.

Now back to our original question about finding the area of a rectangle. We had already solved the problem of dimension, of how to relate the repeated addition of a line to itself to the designation of an area. We were left with the problem of the unit. Descartes has found a way around the problem of not having a natural unit in lines. He's developed a procedure for finding the area whether or not the two lines encompassing the shape are commensurable or not. His procedure involves making an arbitrary unit from one of the lines involved in the problem, connecting the two lines in the shape of a triangle, and using proportions to find the result. So far, so good.

I'd be willing to bet, though, that almost no one in at least two hundred years has actually gone through this technique in order to multiply length by width. There is another step we need to get us to where we began, with the multiplicity of ways of multiplying that I learned in grade school and beyond. This further step is provided by the German mathematician Richard Dedekind, an author our juniors will be studying. I'm not going to go into the details here, because they are beyond the scope of this lecture. In short, Dedekind developed an idea of number that went beyond counting numbers to a filled-in *number line*, allowing any magnitude to be represented with digits of some sort, with numbers after a decimal point going on as long as necessary. The square root of two can be expressed as 1.4... pi can be expressed as 3.14... and so on.

Once Dedekind's work led to the number line, and a magnitude was refigured as digits, it was only a matter of time before what we used to call simply "numbers" would come to be referred to as "whole numbers" or "counting numbers," to distinguish them from "irrational numbers," "transcendental numbers," "repeating decimals," et cetera. When we represent magnitudes with digits, the process of getting to an area can be simplified from Descartes' triangles, ratios, and equal rectangles, to a simple technique for decimal multiplication.

For example, (Figure 31) I can multiply one hundred seventy-three by thirty-six and pi by the square root of two in a very similar way. I just change pi and the square root of two into an approximation from the number line, approximations that are developed through other processes that we won't go into here. I ignore the decimals for the moment, multiply each digit together as a number being taken as many times as there are units in the other number, and add them all up. I can then replace the decimals according to rules developed in a sensible way and I have my result. Solving the two problems happens in ways that are nearly indistinguishable.

We've come full circle: geo-multiplication can be performed using digits, with a similar process to numerical multiplication, with no awkward triangles needed. We just do the multiplication and move the decimal place around to where it belongs. The methodological distinction between geo-multiplication and numerical multiplication has vanished. Descartes' claim that what I've been calling geo-multiplication is "the same as multiplication" seems to be vindicated.

But what would I say if someone asked me what I was doing in each case, what I meant by "multiplication" for each problem? In the first case, I think I'd say that I'm taking the first number the number of times that there are units in the second number. Rather than doing thirty-five addition problems, I'm using the much quicker method shown on the slide, but the concept is the same. In the case of π and the square root of two, the technique is the same, but what I'm doing is far more complicated and is really not the same thing at all. I've hidden the distinction and a significant amount of mathematical theory under the simple technique that looks so similar.

Is the process Descartes outlines really "the same as multiplication" as he says? If what we mean by multiplication is finding the answer to a problem, using the technique that we call multiplication, then yes, I'm doing essentially the same thing when I multiply whole numbers as I am when I multiply magnitudes. If what we mean by multiplication, though, is taking the multiplicand the same number of times as there are units in the multiplier, the answer must be "no." It is true that Descartes comes up with a representation of the area of a geometric figure that can be spoken of in terms of a length. He does this, though by declaring an arbitrary unit for each problem. That term, "arbitrary unit," would be an oxymoron if we were following along with Euclid. Euclid's unit, "that according to which each of the beings is called 'one,'" seems not to be arbitrary at all, but to be closely tied to the integrity of the thing in question, as we've said.

Descartes has avoided the use of units, and given us a tool that can find areas with geo-multiplication. This tool has been developed by later mathematicians to the point that literally a child can learn how to use it. As a society, we put a high store on learning "how to do math" by which we mean, in part, how to multiply decimals. The number line is introduced to five- and six-year-olds. It is truly marvelous that things that seemed insurmountably difficult a few hundred years ago are now routinely taught to kids. (Figure 32 – blank)

At one level, I am all for simply appreciating the power of geo-multiplication. I'm in awe, as much as anyone, of the beauty and simplicity of mathematics. I also love me a good problem-solving technique. On the other hand, I can't shake this gnawing suspicion that we have hidden something important, namely, the importance of the distinction between a natural unit and Descartes' arbitrary unity. The distinction between numbers and geometrical magnitudes has been obscured. What we think of that obscurity might relate heavily to what we think mathematics is. I don't know if I had ever asked myself what mathematics is before I got stuck on the problem of multiplying magnitudes. Since I started asking the question, I've heard a lot of responses, which tend to fall into three broad categories: mathematics as a useful tool, as a body of non-contradictory knowledge, and as our access to the truth. I'll take each of these in turn.

Mathematics as a Useful Tool

A tool is a device that achieves a particular goal. It is usually specific to the task at hand, such as a hammer, that is designed to beat a nail into a plank of wood. Viewing mathematics in this way highlights that aspect of tricks, techniques, and methods. Depending on the question I'm asking or the need I'm facing, I'll use a different method. If I'm putting up a zip line in my back yard, I may need some trigonometry to make sure I'm getting the right angle of descent, so I'm neither zipping too fast nor too slow. Mathematics as a tool is a means to an end. It is subservient to whatever the problem is that I'm trying to solve.

Taken more broadly, mathematics can be used not just to solve immediate problems, but to give an account of natural phenomena. Much of the math that we do on the program falls into this category, more or less, of finding the mathematical reasoning that will give an account of phenomena at hand. Galileo wants to understand motion, Euler wants to describe a vibrating string, Bohr wants to penetrate the atom, et cetera. In general, when mathematics is used as a tool for understanding phenomena, the goal is to have an account that can be verified, usually through predicting the phenomena. If I describe the motion of a vibration in the way Euler does and the plucked string follows that motion, I can be confident that my tool is working well. If not, I can go look for another technique.

There's an *ad hoc* character to this description of math, though, that seems to imply too much independence of the techniques from each other. There is no essential connection between tools designed for different purposes. A hammer and

a screwdriver both fit my hand, but other than that, they aren't really related to each other; they don't depend on each other. Techniques of mathematics are not independent in the way a hammer and screwdriver are. Differential equations, calculus, algebra, and geometry all fit into each other. Euler's differential equations and Bohr's atomic equations both rely on algebraic manipulation. In this way, different kinds of mathematics are like countries on a continent rather than tools in a toolbox. Each one is delimited by boundaries, but all go together in an underlying logic.

There is another way in which the metaphor of mathematics as a tool breaks down. Consider the hammer and the nail. Independent from a hammer, a nail is a human artifact. It could be considered a tool itself, a device constructed to achieve a certain end. However, when we use mathematical techniques to build an account of nature, we are not working with one human invention to manipulate another human invention, like we are when we use a hammer to hit a nail. Nature is beyond us, not made by humans. When we examine individual parts of it, we are pulling out a piece of what we imagine to be a whole. It would be strange to approach pieces of a natural whole with mathematical techniques that have nothing at all in common with each other or, worse, were mathematically contradictory to each other.

We see, then, that the relationship between the different branches of mathematics makes some sense if we are thinking of it as a way of understanding nature. We don't want our understanding of nature to be disparate and particulate, so we likewise don't want our mathematics to be thoroughly partitioned into incompatible realms. This leads us to our second way of thinking about mathematics.

Mathematics as a Non-Contradictory System

This account emphasizes the logical dependence of each branch and method of mathematics upon the other and upon some underlying way of thinking. As we noted, the mathematical methods used for each situation are not completely independent from each other, but form a larger body of study. Aristotle, in the *Organon*, maps out syllogistic reasoning in detail, the kind of reasoning used for geometrical proofs and, similarly, in algebraic manipulations. This is the kind of reasoning that deduces true statements from other statements. For example, if we say that all men are mortal and Socrates is a man, we can deduce that Socrates is

mortal. Synthetic knowledge begins with statements from which other statements are derived in a logical system. If line AB is neither less than nor greater than line BC, then it must be equal to it. If force equals mass times acceleration, then half of force also equals half of mass times acceleration.

It is critical that none of the deductions is in doubt. If I determine that AB is not less than BC, I cannot then say that it must be equal to it. There is another possibility to rule out: it might be greater than BC. Likewise, I cannot change the equation in such a way that the equality doesn't hold in all cases. If I'm completely certain about all of my deductions, then I end up with an edifice of statements, each one leading to, stemming from, or interlaced with the others. I can begin with any statement of geometry or any version of my equation and derive many other things.

The question is, where do those original statements come from? For a geometrical system, we want to know where the set of definitions, postulates, and common notions comes from. For an analytic or algebraic system, we want to know where the original equations and the rules of operations come from. For example, in geometry, how do I know that equals, when added to equals, produce equals? It seems like this must be true, but is it simply that I can't imagine the exception? The original statements for any mathematical system may simply be human constructs or symptoms of human limitations of imagination or intellect. Or, they may be true simply, connected with physical or logical necessity.

If we don't know where the original statements come from, we come to think of our mathematical edifice as floating in mid-air, so to speak. It could be self-contained and logically consistent but have no foundation outside of our imagination. Do we have any way of judging the truthfulness of the original statements? Is it possible that our edifice of mathematics is simply contained within my mind, based on foundations that may or may not be true?

Euclidean geometry was the only game in town for centuries. The definitions, postulates, and common notions were toyed with and reworked, but were essentially accepted as true and foundational in themselves. In the nineteenth century, Lobachevsky famously rejected Euclid's fifth postulate, which states that lines that are angled towards each other at less than two right angles will eventually meet. He subsequently showed that the geometry that followed from this rejection was self-consistent and robust. What are we to make of the fact that

two systems of mathematics with different fundamental assumptions are both self-consistent?

Mathematics as Our Access to Truth

In the first possibility I outlined, math as a tool, I was sketching a portrait of a very external process. The mathematician has some problem to solve or phenomenon to describe, and wants something to help with that specific matter. The result was an ad hoc and disparate view of mathematics, a toolbox with different and unconnected tools, each one useful for a specific task.

In the second possibility, math as a self-consistent system, my sketch was very internally focused. The mathematician has a few foundational principles and makes deductions and connections to produce a large mathematical edifice. The result was an integrated whole that may or may not relate to anything outside itself. Neither possibility is completely satisfactory. The handy man or woman might be satisfied with using math as a tool, but at some point, someone will ask how the tools are connected. The student of mathematics might be fascinated by the beautiful connections between theorems and equations, but at some point, someone will ask what the equations are good for.

To connect the external and the internal, the foundational principles of mathematics must be grounded in reality. Mathematics must be seen not as useful simply by chance, not just self-consistent in the mind, but approaching something true. As Pontius Pilate famously asks, “What is truth?” It’s unclear from the *Gospel of John* whether Pilate’s question is genuine or dismissive. The claim that there is a truth to things and that we can approach this truth is a very large and interesting claim, and underlies the further claim that mathematics is one of the primary ways we have to approach it.

There are several authors in the program who seem to hold versions of this view. I’ll outline three approaches here, presented under the names of authors who seem emblematic of each approach, namely, Plato, Galileo, and Kant.

Plato

For the purposes of this lecture, I will treat Socrates as something of a mouthpiece for Plato, the author. Socrates’ project, as it plays out in the dialogues, is a search for wisdom. He looks for wisdom among men, through conversation. He questions people who are willing to discuss fundamental things until the interlocutors come

to an agreement; about the meaning of some important term, such as virtue or justice. If the interlocutors can discard contradictory notions, they will be getting closer to an expression of what virtue or justice is, for example, even if they never get to a full definition. Language, then, reflects reality by giving us a way to the invisible forms which give solidity and integrity to the world.

In the *Republic*, Book VI, Socrates uses the image of a divided line to help the interlocutors understand how to approach this invisible structure of reality. (Plato, p. 190) Part of the line is concerned with the physical world, and is divided into two parts, one corresponding to the natural things and the other to the images of natural things. The other part of the line is concerned with the invisible world, which is only approachable through the intellect. It is divided into two also, one part corresponding to the forms and the other to geometry and calculation. If the analogy holds between the two different regions of the line, mathematical objects would relate to the forms themselves similarly to how images and shadows relate to objects in the world. (Figure 33) In Socrates' language, the geometers, in their use of drawings of geometrical shapes, are "...are really seeking to behold the things themselves, which can only be seen with the eye of the mind." (Plato, p. 191) This exercise gives them a route into the invisible reality of things. The more one studies geometry and calculation, the closer one would be to understanding the forms.

Perhaps we start to discern the kernels of excellence in human activity similarly to finding the form of a triangle in the slope of a zipline between two trees. Our minds become accustomed to sifting out contradictions, so that our gaze is sharp enough to see what justice is, with the right interlocutors.

Galileo

To set the stage for Galileo, we need to acknowledge a shift from the pagan philosophers and thinkers, such as Euclid, to those rooted in Judeo-Christianity. The Hebrew book of Genesis tells the story of creation, ending the sixth day with the creation of man and woman, made in God's image and with his likeness. This account of the world as a created thing, coming from one mind, was central to Hebrew thought and was later spread widely through Christianity. Reality, in Judeo/Christian thought, does not consist in separate forms which give integrity to the world. Rather, the world has come to be through the power of one mind,

through the God of the universe, who brought everything, including humanity, into being by speaking it into existence.

Johannes Kepler, (Figure 34) who lived around the same time as Galileo, says in Chapter 7 of his *Astronomia Nova*, “It is true that a divine voice, which enjoins humans to study astronomy, is expressed in the world itself, not in words or syllables, but in things themselves and in the conformity of the human intellect and senses with the sequence of celestial bodies and of their dispositions.” (Kepler, p. 41) In other words, God’s voice is responsible for both humans and the planets. This voice expresses itself in the things themselves, both the astronomical objects and our human senses and minds. We are made to understand the universe.

Galileo must have been heavily influenced by this mindset as he went about his study of motion. (Figure 35) In his essay *The Assayer*, which is quoted at the beginning of our junior laboratory manual, he says, “Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.” (Galileo, p.4)

For Galileo, Kepler’s voice of God shows itself as a philosophical book, the book of the universe that surrounds us. We are part of that universe in a particular way, in that we can understand the language and read the book. The language is mathematics, the letters are the figures of geometry. We can interpret the physical world through math, learning more about reality and, perhaps, about God himself, by reading his book. Mathematics is essential to the whole project and gets us as close as we ever could be to understanding our world.

Kant

Kant came into the conversation after Hume had made a strong case for mathematics as an internal activity of humans, something that does not get us closer to the truth of things, but is that is purely self-contained. Kant takes this idea of internality quite seriously. He works hard to make the connection that Hume had rejected, the connection between the internal and external, between mathematics as a mental activity and mathematics as an approach to truth. He believes that he does bridge this gap, though how he does it is difficult to say, though our juniors will be

taking a stab at it next winter. One thing Kant doesn't do is claim that we can achieve knowledge of the thing in itself through our senses, reason, or mathematics. (Figure 36) In the Prolegomena, he says, "And we indeed, rightly considering objects of sense as mere appearances, confess thereby that they are based upon a thing in itself, though we know not this thing as it is in itself, but only know its appearances, viz., the way in which our senses are affected by this unknown something." (Kant, paragraph 32) The "thing in itself", in his language, is always out of reach for us. If by mathematics approaching truth you mean getting through all the layers to the essence of what something is, Kant would not agree with that.

Still, he does claim to put math on a firm foundation, saying that we can approach the world and find objective truths in it. Mathematics, then, is both internally and externally facing for Kant. Our own internal mathematical apparatus, filled in and broadened out, is our way of working with things in space and time. We may not be able to get at the thing in itself--we cannot truly bring the external world inside of us or export the internal descriptions to the outside--yet we do truly interact with the external world in some way, which involves the use of mathematics.

As we encounter the world, the unapproachable noumena in the world and the apparatus of mathematics in the mind come together as phenomena. Encountering elements of the world simply as phenomena, we can grasp things, visualize them in motions that follow certain laws, and describe those laws mathematically. The project of understanding the natural world that was so important to Galileo can continue, if, as Kant would have it, we understand that we are not really getting at the reality of things in themselves. (Figure 37 – blank)

Conclusion

What has happened to my original question about multiplication? Recall that we had gotten to the point of wondering what material difference eliminating a sense of a natural unit would make. Why not simply applaud Descartes and move on?

If mathematics is only to be used as a tool and nothing else, I don't think it matters much. We use mathematical operations such as multiplication as tools in the way we want to, not worrying about whether the tools connect with each other. If mathematics is an internally consistent system, it matters more whether our units

are arbitrary or natural. The easiest way to reconcile the two is to say that units are arbitrary and numbers are best represented on the number line. It doesn't really affect how I see the world around me, since this premise relies only on having an internally consistent mathematical system. Whether or not there is any connection to something real in the world is beside the point.

What if I believe that mathematics is a, or perhaps the, way into learning the truth about things? This is the case where it matters most whether I call my units "that according to which the beings are called one" or not. If my way of understanding the world comes through mathematics, and if I've said in my mathematical system that units are simply arbitrary, then I might at least wonder if the beings, the things that are, have any kind of integrity to them. If I prefer the Kantian approach, I can always hold onto my belief that any such thing in itself has integrity, is a real thing, even if my mathematical work relies upon arbitrary units.

For the Platonic and Galilean approaches, would I assume that because I have come to the point of using the number line as a reflection of what is real, what I call a thing is arbitrary? In other words, if I've left off using natural units, does that say that nature itself has no units? If I have other reasons for wanting to say that nature does have units, that there are beings with their own integrity in the world, what do I do with a mathematics that doesn't recognize this anymore?

I began this lecture with a naive curiosity about how multiplication came to be used with lengths rather than with numbers. My mother's Euclidean explanation of multiplication morphed over the years of my education, under the layers of learning how to perform operations and when to use which mathematical tool. I, myself, am persuaded that math is not simply a tool, nor is it simply a self-consistent system, unconnected with what is real. I'm left, then, with this final question: If mathematics really is a route into what is true, what is it telling us about our world?

FINIS

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