LOGICAL ECONOMY IN EINSTEIN'S «ON THE ELECTRODYNAMICS OF MOVING BODIES»

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...Time is come round,
And where I did begin, there shall I end.

In section 2 of his 1905 paper[1] Einstein shows that the synchronized clocks of the "stationary" system K can be coordinated with the clocks of the moving system k only at the cost of violating the criterion of simultaneity in k. Thus, if the clocks in k are to be synchronized (as well as those in K), we must abandon the attempt to coordinate the two systems according to the classical formula $\tau = t$. The task of section 3 is the derivation of new transformation equations which will allow observers in K to coordinate the measurements (temporal and spatial) of observers in k with their own. Something more than midway in the argument the equations take the form

$$\tau = \varphi(v)\beta(v)(t-vx/c^2),$$
$$\xi = \varphi(v)\beta(v)(x-vt),$$
$$\eta = \varphi(v)y,$$
$$\zeta = \varphi(v)z,$$

where $\beta(v) = 1/(1-v^2/c^2)^{\frac{1}{2}}$. [2]

Hereafter I shall refer to the conjunction of the above equations as T

1. All page references are to the English translation in Einstein et al., *The Principle of Relativity* (Dover) and, when in brackets, to *Das Relativitätsprinzip*, ed. Blumenthal (Teubner).

2. I shall use $\beta(v)$ instead of Einstein's β in order to stress the dependence of β on the values of v and c (the velocities of k and of light) as *measured in K.*

The next stage in the derivation is the evaluation of $\varphi(v)$, which will conveniently assume the value 1 and thereby absent itself from T. Einstein, however, does not press on directly to that mark. Instead, he says,

> We have now to prove that any ray of light, measured in the moving system, is propagated with the velocity c, if, as we have assumed, this is the case in the stationary system; for we have not as yet furnished the proof that the principle of the constancy of the velocity of light is compatible with the principle of relativity. (p. 46 [33])

His proof then takes the form of showing that if $x^2+y^2+z^2 = c^2t^2$ ("as we have assumed"), we shall arrive, by application of equations T, at the result $\xi^2+\eta^2+\zeta^2 = c^2\tau^2$. [3]

The proof is puzzling on two counts. First, why does Einstein place it where he does, interrupting his otherwise single-minded thrust toward the final statement of equations T with $\varphi(v) = 1$? But, more importantly, does not Einstein here profess to *prove* a proposition ($\xi^2+\eta^2+\zeta^2 = c^2\tau^2$) from

3. One is initially tempted to reverse the order of the proof, since equations T may be directly applied to $\xi^2+\eta^2+\zeta^2 = c^2\tau^2$ to yield $x^2+y^2+z^2 = c^2t^2$. For reasons which will soon be manifest, the longer haul of the 1905 paper is requisite. Solving for x, y, z and t, we have

(a) $t = [\tau/\varphi(v)\beta(v)] + vx/c^2,$

(b) $x = [\xi/\varphi(v)\beta(v)] + vt,$

(c) $y = \eta/\varphi(v),$

(d) $z = \zeta/\varphi(v).$

Solving (a) and (b) simultaneously, we have

(a') $t = [\tau\beta(v) + (v\xi/c^2)\beta(v)]/\varphi(v)$

(b') $x = [\xi\beta(v) + v\tau\beta(v)]/\varphi(v).$

The functions $\beta(v)$ and $\varphi(v)$ continue to refer only to measurements made in K; however, when (a'), (b'), (c) and (d) are applied to $x^2+y^2+z^2 = c^2t^2$, $\beta(v)$ and $\varphi(v)$ cancel out, leaving the equation free of all coordinates in K.

premises T which themselves were derived from that very proposition? For, after deriving a provisional equation for τ (p. 45, line 3 [32, line 24]), he determines the quantities ξ, η and ζ on the basis of the assumption that "light...is also propagated with velocity c when measured in the moving system," i.e. that $\xi^2+\eta^2+\zeta^2 = c^2\tau^2$ (or, as Einstein employs it, that $\xi = c\tau$, $\eta = c\tau$ and $\zeta = c\tau$ for rays propagated along the ξ, η and ζ axes respectively--p. 45, lines 8-12, 14, 23 [p. 23, lines 28-31, 33, and p. 33, line 8]).

The latter difficulty can be schematized as follows. Let $\underline{K}$ represent the equation $x^2+y^2+z^2 = c^2t^2$ and $\underline{k}$ represent the equation $\xi^2+\eta^2+\zeta^2 = c^2\tau^2$. Also, let S stand for other propositions which shall hereafter be identified. Then we seem to have the following pair of arguments:

$\underline{K}$ & $\underline{k}$ & S $\rightarrow$ T (pp. 44-46, line 6 [p. 31, line 37-p. 33, line 15])

T & $\underline{K} \rightarrow \underline{k}$ (p. 46, lines 11-29 [p. 33, line 20- p. 34, line 3]).

In short, the argument is circular: $\underline{k}$ is proved by means of T, which itself is proved by means of $\underline{k}$. Not all circles, however, are vicious. To show that the circle which Einstein's argument defines is a noble one, I must turn to the more fundamental considerations of section 2.

PART I: A MATTER OF PRINCIPLE

1. On the Two Principles

At the beginning of section 2 Einstein sets forth the two principles upon which his subsequent reflections are based and which I shall henceforth refer to as P_1 and P_2:

P_1: The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.

P_2: Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or a moving body. Hence

$$\text{velocity} = \frac{\text{light path}}{\text{time interval}}$$

where time interval is to be taken in the sense of the definition in § 1.

By calling them principles (*Prinzipien*, ἀρχαί), Einstein indicates that P_1 and P_2 supply radical beginnings which will rule over our subsequent thinking. That is to say, neither can be derived from the other (for in that case the derived proposition would not be a principle), nor can either be omitted if the subsequent propositions are to follow.

We note two pecularities in the phrasing of P_2. Why does Einstein surround the word "stationary" with quotation marks (see also p. 43, line 7 [p. 31, line 7]), whereas he never refers similarly to a "moving" system? Further, why does he refer to "*the*" stationary system (*im "ruhenden" Koordinatensystem)*? Ought he not to have phrased P_2 so as to extend to "any" or "all" inertial systems? (5)

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5. As does, for example, Max Born in his rephrasing of P_2 in *Einstein's Theory of Relativity*, p. 232 (Dover): "In all inertial systems the velocity of light has the same value when measured with rods and clocks of the same kind." Similarly: P. W. Bridgman, *A Sophisticate's Primer of Relativity*, pp.113f. (Wesleyan, 1962) and Hans Reichenbach, *Space and Time*, 204f. (Dover, 1958). Both Bridgman and Reichenbach point to certain logical flaws in the 1905 paper which would have ceased to appear as flaws had the authors treated them as occasions to re-examine the second principle, P_2.

Let us toy with the suspicion that Einstein meant us to take his words literally and see where that suspicion leads us.[6] The word "the" (cp. "*im*") would seem to restrict the claims of P_2 to a single inertial system. To what system? According to section 3, to K. But what is K? Cannot K be *any* inertial system, whose occupants are at liberty to regard their system as stationary? But will that assumption be justified for all inertial systems? Not on the basis of P_2 alone, if "the" is understood literally. Indeed, P_2 would not be incompatible with a world in which there was one system alone in which light was propagated with velocity c regardless of the motion of the source.

P_2, therefore, restricts its claim to the single reference system K. P_2 makes no claims about the behavior of light in other systems, wherein light might travel at velocities varying with the velocity of the emitting source or at some unvarying velocity c' with $c' \neq c$. P_2,, therefore, makes no claims concerning either the existence or the non-existence of an absolute frame of reference. If such an absolute frame exists in the form, say, of a luminiferous aether, P_2 would assert that K is at rest in the aether, in which all observers, regardless of their state of motion, would have grounds for referring to K as *the "stationary" system*.

Lest the above reading of P_2 be thought to rest on an illegitimate overemphasis on a single word, I add three considerations. First, if P_2 is understood to apply immediately to all inertial systems, the proposition <u>k</u> will follow from P_2 alone. However, when Einstein first introduces <u>k</u>,

6. That men of divine intellect wrote with care seems to me as plausible as Ptolemy's assumption that the visible gods Hermes, Aphrodite, Ares, Zeus and Chronos do not wander randomly. In this essay I shall emulate the Ptolemaic canon of interpretation: to save all the phenomena with the least possible number of epicycles.

he offers as justification the premises P_1 and P_2 taken "in combination" (p. 45, lines 7-12 [p. 32, lines 27-31]). That is, the extension of P_2 to systems k other than K requires the additional support of P_1 and is not immediate.

Second, let us assume for a moment that P_2 does not limit itself to K. There is then at least one other inertial system k, in motion relative to K, for which P_2 professes to hold. In this case, P_2 can be analyzed into the two propositions

(a) Any ray of light moves in K with velocity c whether the emitting source be in motion or rest in K,

and (b) Any ray of light moves in some system k', in uniform motion relative to K, with velocity c whether the emitting source be in motion or rest in k'.

But the conjunction of P_1 and (a) implies (b). (b) is therefore *logically redundant* and must away if P_2 is to be a *principle* properly so called.

Third, let us formulate a generalized version of P_2 as follows:

P_2' Any ray of light moves in *every* inertial system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or a moving body. Hence

$$\text{velocity} = \frac{\text{light path}}{\text{time interval}}$$

where time interval is to be taken in the sense of the definition in § 1.

We designate the definition of time interval (to be examined presently) by D. P_2' implies D, since D in imbedded in the very meaning of P_2'. Further, P_2' provides a sufficient justification for D (i.e. $t_b - t_a = t'_a - t_b$) for *any* inertial system--see p. 40, line 12 [p. 28, line 33]. D is therefore *logically superfluous* and section 1 might have been replaced by a deduction of D in the course of section 2.[7] But P_2' perpetrates an even graver superfluity.

7. So, too, does D follow from P_2, *but only for K*. More on this later.

For from P_2' alone[8] *Einstein might have deduced the entire kinematical part of his paper.* Thus, in the first part of the paper, P_1 would have been logically superfluous. This is not all, for the content of the second, electrodynamical part is deduced from the application of T to, e.g., the Maxwell equations, which are themselves logically independent of P_1. To use Kantian terms, P_1 would serve no *constitutive* purpose which was not already adequately provided for by P_2'; at best P_1 would perform a *regulative* office, recommending that Maxwell's equations be generalized under the aegis of P_2'. But to recommend that the laws of physics be reformulated so as to be conformable to P_2' (and its consequent T) is nothing more (and a good deal less) than to assert P_2' ! Thus, *the assertion of P_2' carries with it the assertion that P_1 is logically superfluous in the entire 1905 paper.*

We are forced to the conclusion that P_1 and P_2 can be *principles* only if P_2 is taken literally, as referring to a single "stationary" system K. P_2 is therefore logically neutral on the question of the luminiferous aether. P_2 makes the claim that we can ignore the velocity of emitting *sources* in at least one inertial system K; it makes no claim as to whether or not we can ignore the relative motions of *observers* in systems other than K. Thus, P_2 and the Michelson-Morley experiment (hereafter, MME) are logically irrelevant to one another, for the latter offers no evidence regarding K, which would on their hypothesis be at rest in the aether; the significance of MME depends upon the assumption

8. More precisely, from P_2' and the axioms of Euclidean analytic geometry (E). We may now identify S above with the conjunction D & E.

that the earth (to the second approximation) belongs to an inertial system k ($\neq$ K) concerning which P_2 is silent. But if MME is *logically* irrelevant to P_2, it is *factually* irrelevant to P_1, for the null result of MME at best supplies a *nihil obstat* for P_1. Thus, logical considerations lead to the same conclusion as does Gerald Holton's scholarly argument that the "crucial experiment" of Michelson and Morley was not so crucial after all.[9]

I conclude that P_2 makes a restricted claim, conceivably subject to empirical support, and that P_1 expresses a constitutive, as well as regulative,[10] requirement of pure reason.

2. The Circle Widens

Earlier, I sketched Einstein's circular argument as

$$\underline{K} \;\&\; \underline{k} \;\&\; S \longrightarrow T \qquad \text{(with S = D \& E, note 8 above)}$$

$$\text{and} \quad T \;\&\; \underline{K} \longrightarrow \underline{k}.$$

Several observations are now in order.

First, $\underline{K}$ is directly implied by T (assuming the axioms of E), for T contains the equation $\beta(v) = 1/(1-v^2/c^2)^{\frac{1}{2}}$; that is, T contains an equation which asserts that $1/(1-v^2/c^2)^{\frac{1}{2}}$ is a function of v alone. In other words, c has a constant value (measured in K--cf. note 2) regardless of the velocity v (also measured in K) of its source. (This should not surprise us, for T was deduced from $\underline{K}$ and hence from P_2.) Thus, the second half of the circular

9. *Thematic Origins of Scientific Thought*, II, 9 (Harvard)

10. Einstein also refers to his two principles as *Voraussetzungen*, "presuppositions" ([p. 26, lines 28, 29, 32] cp. p. 38, lines 3, 4, 8). Although *Voraussetzung* has the primary sense of a (constitutive) presupposition, its meaning may be extended to that of a *goal set forth in advance* so as to direct our activity. *Voraussetzung* thus plays, in our thinking activity, a role analogous to Aristotle's τέλος, which he identifies with the ἀρχή which *is*, i.e. which both constitutes the being of things and regulates their development.

argument reduces to $T \to \underline{k}$. We are further justified in asserting $T \to \underline{K}$ & $\underline{k}$.

Second, $\underline{k}$ is asserted to hold for *any* inertial system k. $\underline{k}$ is therefore equivalent to the proposition that the law expressed in P_2 as holding for the system K may be extended to any system k in uniform motion relative to K. But that extension is guaranteed by P_1. Let us analyze P_1 into two parts:

P_1' *If* P_2 is a law holding true for the system K, P_2 will hold true for any other inertial system k;

P_1'' Any law other than P_2 by which the states of physical systems undergo change is not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.(11)

Now, it is precisely P_1' that allows the extension of P_2 (which is in itself restricted to K) to any inertial system k. Hence, $\underline{K}$ & $\underline{k} \Leftrightarrow P_1'$ & P_2. Therefore, $T \to P_1'$ & P_2.

Third, we turn to the first half of the circular argument: $\underline{K}$ & k & $S \to T$. Since $\underline{K} \Leftrightarrow P_2$ and P_1' & $P_2 \to \underline{k}$, we may write P_1' & $P_2 \to \underline{K}$ & $\underline{k}$. Therefore, P_1' & P_2 & $S \to \underline{K}$ & $\underline{k}$ & $S \to T$. Einstein's circular argument now expands into

$$P_1' \,\&\, P_2 \,\&\, S \to T$$

$$\text{and} \quad T \to P_1' \,\&\, P_2.$$

The question of circularity now appears--as Einstein's own words already inform us--to be a matter of *principle*.

11. The separation of P_1 into P_1' and P_1'' seems artificial, but I see no reason to question its validity. It is essential that P_1' take a hypothetical form and that P_1 not be understood to contain implicitely within itself the laws of physics of which it speaks. Otherwise, all the laws of physics, including P_2 (if P_1 and P_2 are both true), could be deduced from P_1 alone, and P_2 (whether true or false) would lose its right to be regarded as a principle.

3. Circularity and Consistency

A widened logical circle is nonetheless a circle and, ordinarily, a defective piece of argumentation. To meet this objection we must first of all observe that in his seeming 'digression' of p. 46 (i.e. the argument $T \to \underline{k} \to P_1$ & P_2), Einstein makes no claim to have the *truth* of $\underline{k}$ or, for that matter, of P_1 and P_2. His object, twice stated, is to prove that his two grounding principles (*Grundprinzipien*) are combinable (*vereinbar*) with one another (p. 46, lines 14-16, 28f. [p. 33, lines 22-24; p. 34, lines 2f.]). What is at stake is the mutual *consistency* of P_1 and P_2.

Now, is there any reason to suspect that P_1 and P_2 are logically incompatible? I think the answer must be "Yes". For the conjunction of P_1 and P_2 leads to the highly paradoxical conclusion--easily stomached only by those of us who have been weaned on the 1905 paper and its later endorsements--that the velocity of light in free space will appear to have the same value for observers in different inertial systems. The paradoxicality of such a claim led Lorentz in 1895 to rescue the Michelson-Morley project with the counter-claim that bodies undergo a shortening in the direction of their motion through the aether. If "P_1 & P_2" can engender so monstrous a pair as "$\underline{K}$ & $\underline{k}$", perhaps further reflection will derive the canonically contradictory pair "p & not-p". The burden of proof rests with Einstein.

Bearing in mind that the issue is consistency, not truth, we now make a brief observation on the relation of circularity to consistency. Let Q and R stand for two collections of propositions. The purest (though not the only) instances of circular argumentation would take the form

$$Q \to R \quad \text{and} \quad R \to Q$$

where R's only title to credence is its derivability from Q. The argument form therefore fails to establish the *truth* of Q, for it reduces to the trivial assertion $Q \rightarrow Q$. However, with the question of *consistency* a different outcome is possible. Let us suppose that the consistency of Q is in question and that there is no direct proof at hand of Q's consistency or inconsistency. Suppose, however, that such a proof is forthcoming for R (whose only claim to plausibility derives from Q). Then, from the demonstrably consistent propositions R we would have deduced propositions Q. *But from a consistent group of propositions (R) no contradiction can follow.* If Q were inconsistent, it would follow that a contradiction lurked, however obscurely, in R. Hence the consistency of R implies the consistency of Q. In other words, though a circular argument cannot establish truth, it can, under certain conditions, establish consistency.

4. The Circle Vindicated

Our task now takes the form of showing that T is consistent. For, if T is consistent and if $T \rightarrow P_1' \ \& \ P_2$, it follows that P_1 and P_2 are mutually consistent. Is there any direct way in which the consistency of T may be tested?

Let us recall the constituents of T. They are the equations

$$\tau = \varphi(v)\beta(v)(t-vx/c^2),$$

$$\xi = \varphi(v)\beta(v)(x-vt)$$

$$\eta = \varphi(v)y,$$

$$\zeta = \varphi(v)z,$$

$$\text{where } \beta(v) = 1/(1-v^2/c^2)^{\frac{1}{2}}.$$

We must show that each of the equations T is consistent with itself and consistent with the others.

First, the question of self-consistency. We shall assign only real number values to τ, ξ, η, ζ, $\varphi(v)$, t, x, y, z and, of course, to c. Also, we shall restrict v to real number values which satisfy $v^2 < c^2$; hence $\beta(v)$ is restricted to the real values $1 \leq \beta(v) < \infty$. Thus, the right-hand members of the five equations take on real number values in all allowable cases. The left-hand members of each equation taken singly do not appear in the right-hand expressions; therefore, each left-hand member is free to accomodate itself to the (real) values assumed by the right-hand members. Each equation is therefore self-consistent.

Next we turn to the question of the mutual consistency of the five equations. First, we note that the fifth equation could have been done without: we could replace $\beta(v)$ with $1/(1-v^2/c^2)$ in the first four equations (with the restriction $v^2 < c^2$). The question now reduces to the mutual consistency of the first four equations. The right-hand members are, as we saw, limited to real values. But the left-hand member of each equatinn appears *uniquely* in that equation. Thus, the left-hand members of each of the first four equations can severally and *in complete independence from one another* absorb the shock of any allowable assignment of real values to the right-hand members. Therefore, the equations T are logically independent and mutually consistent.

Since T is consistent, we conclude that P_1' and P_2 are consistent. Einstein, however, makes a broader claim: that P_1 and P_2 are consistent. Now, it is certainly not the case that $P_1' \ \& \ P_2 \rightarrow P_1 \ \& \ P_2$, since P_1 contains the indefinite proposition P_1'' as well as P_1'. However, we analyzed P_1 into P_1'' and P_1'' precisely with a view to isolating that part of P_1 which is logically relevant to P_2. P_1'' is therefore what is left of P_1

after the latter has been stripped of all logical relevance to P_2. Therefore, the conjunction of P_1'' with P_1' and P_2 produces no logical inconsistency. Since the (proved) consistency of T implies the consistency of P_1' & P_2 to which we may without inconsistency conjoin P_1'', we have

T is consistent,

and T is consistent $\longrightarrow$ P_1' & P_1'' & P_2 are consistent
$\longrightarrow$ P_1 & P_2 are consistent,

therefore P_1 & P_2 are consistent,

which is what Einstein claims to have proved.(12)

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12. To the consistency proof given above the objection might be raised that at some time a law L might be discovered which contradicts P_2. Certainly, there is nothing in P_1 to suggest that L is false or, for that matter, true (see note 11 above). That P_1 & L might be consistent with each other and jointly imply not-P_2 merely shows that P_1 is logically independent of L (or *any* law of nature, including P_2).

    Moreover, if the foregoing interpretation is correct, the note at the bottom of p. 46 [p. 34] is unintelligible:

    > The equations of the Lorentz transformation may be more simply deduced directly from the condition that in virtue of those equations the relation $x^2+y^2+z^2 = c^2t^2$ shall have as its consequence the second relation $\xi^2+\eta^2+\zeta^2 = c^2\tau^2$.

    Suffice it to say that the note does not appear in the 1905 paper and is presumably one of the several additions of the annotator A. Sommerfeld. Einstein does not claim that K implies k; what he does claim is that K (which is implied by P_2) *together with P_1* implies k. But this claim is common to the 1905 paper and, for example, the way in which Einstein "more simply" deduces T in his popular exposition of the Special Theory of Relativity (*Relativity: the Special and General Theory*, Appendix I: Crown, 1961). The simpler derivation does indeed beg a question, but not the one which Sommerfeld's note suggests (see the following section of this essay).

* * *

The foregoing analysis leaves unanswered the first of the two questions raised on page 2: why does Einstein place his consistency argument in the middle of section 3, thereby interrupting his derivation of the transformation equations? More important is an objection raised in somewhat different forms by P. W. Bridgman, H. Reichenbach, and L. Sonnenfeld: is not Einstein's argument forced into an even wider and perhaps genuinely vicious circle by his use of D in the form $\frac{1}{2}(\tau_0 + \tau_2) = \tau_1$ (p. 44, line 17)?

I believe the question can be given a partly satisfying answer and that the objection can be met. I hope to set forth my arguments thereto within a few weeks.