Speedy Achilles[1]

Among the many features which make Homer such a delight to read are his epithets. "Rosy-fingered dawn," "wine-dark sea," "loud-thundering Zeus," and "speedy Achilles" are only a few of these ever recurrent, yet never stale, facets of Homeric verse. The scholar Milman Parry saw these stylistic landmarks as a store of fixed formulas, whose readiness-to-hand eases the task of both the singer and the hearer of orally-transmitted tales.[2] Certainly they are not mere adjectives: as the same writer points out, "speedy Achilles" is "speedy" even when he is not running. "Speedy" expresses something of Achilles' very being; the epithet conveys not what he does but what he is.

But if the epithet is not tied to visible activity, doesn't that create a gulf between the epithet and our awareness? The epithets should then feel alien and abstract. But such is not our experience as readers of Homer. We do not seem to require overt manifestations of Homer's epithets. Whatever it is that identifies the Achaeans as "strong-greaved," it does not require a portrayal of their shin-armor. We can, similarly, appreciate the earth-shaking of Poseidon, and the shining helm of Hector, wholly apart from sensuous descriptions.

We all of us have our favorites among these sobriquets. Who doesn't warm to "Odysseus, man of many devices" and "Hector, breaker of horses"? My personal all-time favorite is "Apollo, destroyer of mice."[3] But many years ago I developed a special fondness for "speedy" Achilles thanks to Sam Kutler, then my colleague in Annapolis. Most translators render "swift-footed"; but Sam preferred "speedy," and he continually delighted in contrasting Homeric speed with the *ratio-of-distance-to-time*. I should like to explore that contrast in today's talk. Let us therefore turn our attention, for the moment at least, away from the Homeric world and, instead, listen in on a conversation taking place close by the Venice Arsenal sometime in the early 1600s.[4]

{ Part I. Speed }

Juniors and seniors will recall Galileo's discussion of motion in the Third Day of his *Dialogues Concerning Two New Sciences*. That book recounts the conversations of three friends as they study still another book, the work of a greatly admired, but otherwise unidentified "Academician," whom I will take the liberty of supposing to be Galileo himself, and thereby save considerable circumlocution. As Salviati reads aloud in Latin, the friends discuss propositions concerning *equable motion*, culminating in Proposition VI:

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[1] Lecture given by Howard J. Fisher at St. John's College, Santa Fe, 9 April 2025.

[2] Milman Parry, *The Making of Homeric Verse*, Collected papers of Milman Parry, tr. and ed. Adam Parry. Oxford (1971).

[3] *Apollo Smintheus*: Teukros, son of the river-god Scamander, was said to have built a temple to Apollo Smintheus or Apollo, "destroyer of mice" (lit. "mouse-god") after Apollo destroyed the mice infesting the Troad during Teukros's reign. Hard, Robin. *The Routledge Handbook of Greek Mythology: based on H.J. Rose's "Handbook of Greek mythology*." Routledge (1986). Compare *Iliad*, I.39. I'm sorry to report that nearly all modern translators have rejected this rendering.

[4] Galileo published *Two New Sciences* in 1638.

> If two moveables are carried in equable motion, the ratio of their speeds will be compounded from the ratio of spaces run through and from the inverse ratio of times.[5]

Anyone familiar with Euclid will feel completely at home with this language. We appreciate how conscientiously Galileo forms ratios of *like magnitudes only:* speed to speed, distance to distance, and time to time. He is following the canons of Euclidean reasoning, and we may do so, too, by expressing the relation as

$$v : v' :: s : s' \text{ comp. } t' : t \,.$$

Notice that the ratio of times is *inverted*, as the proposition required: the primed value appears as the *antecedent*, in contrast to the ratios of speeds and of distances, where the primed magnitudes are the *consequents*.

Now if we choose v' s', and t' as the *units* of speed, distance, and time, respectively, we can proceed to *measure* each of those quantities and so replace the *ratios of magnitudes* with *ratios of numbers* in which each of the quantities denoting a *unit* is replaced by the number 1; and the relation will take this form:

$$v : 1 :: s : 1 \text{ comp. } 1 : t$$

from which

$$v : 1 :: (s \times 1) : (1 \times t) :: s : t \,.$$

And since these ratios are, as we said, ratios of *numbers*, the product of the means will equal the product of the extremes; we write:

$$\text{MEANS} \rightarrow \;\; 1 \times s = v\, t \;\; \leftarrow \text{EXTREMES}$$

or, expressed more conventionally,

$$s = v\, t \,.$$

This is the algebraic expression of Galileo's proposition; and I hope you found it useful to see how, without violating any Euclidean canons, the algebraic equation follows directly from the expression in terms of ratios. We often have occasion to contrast the language of ratio with the language of algebra; nevertheless, since nothing in the algebraic equation contradicts Galileo's Euclidean principles, I trust that to express "speed" algebraically, as I have done here, will not depart in any essential respect from Galileo's own thinking.

Symbolic notation does, however, introduce an ambiguity that requires our attention: specifically, the "s" in this equation could denote a particular *position*; but in fact it represents a *distance run through*—a *change* in position, as Galileo's prose statement makes clear. Similarly, "t" does not here represent a *moment* of time, but rather the time *interval* occupied by the motion in question. We can make these associations explicit by means of the symbol Δ, which generally indicates a *change*, a *difference*, or an *interval* of something. Let us therefore rewrite the equation as:

$$\Delta s = v\, \Delta t \,,$$

where Δs denotes the interval of distance traversed by the moving being during Δt, the interval of time. And from this equation, we can state the *measure of speed* as the quotient of *distance traversed* by the *time-interval* required, that is:

[5] Proposition VI of Equable Motion, Stillman Drake translation.

$$v = \frac{\Delta s}{\Delta t}$$

We will return to this expression. But notice that, though we developed this measure of speed for *equable* motion—that is, *constant speed*—nothing in the *definition* specifies whether the speed is constant or not. As a definition of *measure* of speed, then, it should apply in either case.

But what kind of "speed" is this? It is quite different from Achilles' epithet. Achilles' speediness is an *element of being*, an attribute rather than a predicate; let me explain what I mean by this.

Traditional grammar identifies the predicate in a sentence as, basically, everything that is not the subject. *Predicate*, therefore names whatever is asserted of the subject—but *not the subject itself*. The predicate is thus *external* to the subject, joined with it only by *the act of speaking* (L. præ-dicāre, to speak forth). In contrast, an *attribute* typically carries a much closer sense of connection to the noun it modifies. With respect to the subject, an attribute is a "given" (L. tribuĕre) in a way that is not dependent on speech. In its strongest form, the epithet, it is almost part of the noun itself. In particular, it has nothing to do with *time*.

So, too, Achilles' epithet "speedy" is independent of time. By contrast, the measure

$$v = \Delta s / \Delta t$$

involves time explicitly. The Homeric epithet is never separated from Achilles himself; but *the quotient* stands apart. *Measured* speed can only be predicated of Achilles; it is in no way essential to him.

As modern readers, we may easily fail to realize what Galileo has done. So accustomed are we to thinking of "speed" as a quotient of distance by time, that when Galileo's proposition triumphantly brings forth the equivalent of that very quotient as its conclusion, we are likely to deem it either trivial or circular. And so it would be, had Galileo been thinking of speed as a ratio all along. But he was not. On the contrary, Galileo was actually beginning from a far more immediate, even primordial, notion of speed. Such, at least, was Sam Kutler's opinion—that we can best understand the concept of speed with which the Third Day begins by appreciating the epithet of speedy Achilles, and consequently seeing Galileo's work as that of transforming the epithet into a *magnitude*.

What does it take to qualify as a magnitude? The minimum requirement, it seems, is the ability to sustain relations of *greater* and *less*, as Euclid's Definition 1 exemplifies in Book Five:

> A magnitude is part of a magnitude, the less of the greater, when it measures the greater.

Galileo asserts "greater" and "less" of *speed* with these two Axioms:

> Axiom III. The space traversed with greater speed is greater than the space traversed in the same time with lesser speed.

and

> Axiom IV. The speed with which more space is traversed in the same time is greater than the speed with which less space is traversed.

We can be in no doubt that "greater and less" is the essential factor in Achilles' pursuit of Hector in Book 22; for in order to catch up with Hector and slay him, Achilles has to "traverse more space in the same time" than Hector does:

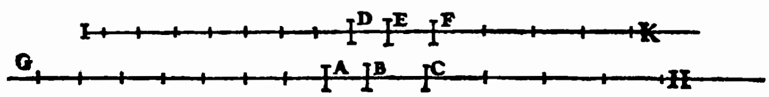
> And the shivers took hold of Hektor when he saw him, and he could no longer stand his ground there, but left the gates behind, and fled, frightened, and Peleus' son went after him in the confidence of his quick feet. ... [S]o Achilleus went straight for him in fury, but Hektor fled away under the Trojan wall and moved his knees rapidly.[6]

By Axiom IV, therefore, the speed of Achilles is greater than the speed of Hector. You see that in respect of being greater and less, the "speed" with which Galileo's axiom is concerned is firmly rooted in the Homeric world.

But Galilean speed is "magnitude" in a stronger sense too: that of being able to sustain *ratio*. We saw this earlier, when Proposition VI expressed the *ratio of speeds* as the compound of the ratio of distances and the inverse ratio of times. I did not question Galileo's ratio language at that point, but now we must ask, what is its foundation?

The first time Galileo speaks of speed having *ratio* is in Proposition II:

> If a moveable passes through two spaces in equal times, these spaces will be to one another as their speeds . . . .



What more, then, besides the capability of being greater and less, does it take for magnitudes to admit of *ratio*? Euclid's Definition 4 of Book Five declares:

> Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

Galileo bestows this capability upon *speed* when he unceremoniously takes "equimultiples" of speeds AB, DE and BC, EF[7]—for multiplication is just a sequence of additions. You can see those equimultiples represented in the figure.

But what gives Galileo the right to assume additivity of speeds? I believe we can find that property, too, in the Homeric world. Not, however, where we might most expect it: that would be in a context like spear-throwing, where a spear thrown *on the run*, rather than *from a stand*, has the advantage of added speed—the sum of the speed of the cast and the speed of the sprinting spearman. But contrary to such expectations, all the spear-throwing in the *Iliad* seems to take place from stationary positions, where the combatants first goad and insult one another, and only then take turns with their casts.

If not in spear-combat, then, where do we find *addition of speed*? A striking incident occurs in the chariot race of Book 23. Antilochos and Menelaos are approaching a dangerously narrow spot on the course; but, instead of restraining his horses, Antilochos

> drove them all the harder with the whiplash for greater speed" (23.429ff).

[6] *Iliad* Book 22, excerpt from lines 139–144 (Lattimore translation). Unless otherwise identified, all subsequent citations of Homer will be from the *Iliad*.

[7] Galileo, Third Day, NE 192

We have an image of Antilochos, with his whip, actually impressing speed into his horses. A similar action was suggested even more strongly in Book 17 when Zeus, pitying the horses of Aiakides,

breathed great speed[8] into them (17.456),

as though *speed* were something that can be transferred from one habitation to another.

Galileo, then, is making explicit elements that were implicit in Homer. In doing so, however, he will convert Achilles' epithet into something that has little place in the Homeric world. I mentioned earlier that Achilles' epithet never departs from him, for it has no connection with time. But the time-dependent magnitude that constitutes Galilean speed is only a predicate, not an attribute. It can be present or absent, can increase or decrease. It cannot escape conditionality and therefore cannot escape time. The nearest a time-bound characteristic can approach to possessing essentiality is—*permanence*. The permanent predication of "speed," defined as a quotient, requires that Achilles be characterized by $\Delta s/\Delta t$ *at all times*—that is, *at every moment*.

Well, we all know the conundrum *that* brings about; but let me recount it. We wish to assert that Achilles has *speed* at a certain time t; but the definition obliges us to specify an *interval* of time, Δt. The only value of Δt that is consistent with "a certain time" is zero! But the only value of Δs that is consistent with Δt being zero is likewise zero—in which case, as Zeno gleefully pronounces, the moving body must be at every moment *at rest*.

Now, as we know, both Newton and Leibniz advanced brilliant techniques to dissolve this conundrum and save the definition. We shall, gratefully, adopt those methods; but notice that in a certain sense, the more successful they are, the more they miss the point; for elements of *being* cannot meaningfully be ascribed to time. We do not say that the greater exceeds the lesser "at a certain time," or even "at all times"; time is simply extraneous to the relation of the greater to the lesser. By converting Achilles' epithet into a *time-linked predicate*, Galileo has begun to wrest Achilles out of his world of being and into a world of happening.

We, in our era, still bear the weight of that extraction. It is no easy task for us to grasp *being* in articulate consciousness. Even Keats' "still unravished bride of quietness" finds only *suspension of action*, not *essence*, in the line

"Bold lover, never, never canst thou kiss."[9]

Nevertheless, we seem to have been willing to pay such a price in return for rescue from the menace of Zeno. Let us therefore make payment in full by considering not only finite but *infinitesimal* ratios of distance and time—for example, by contemplating, with Newton, the *vanishing ratio*: the ratio with which the change in distance, Δs, and the change in time, Δt, together vanish to nothing. Or, with Leibniz, by letting those changes become *infinitesimal*—Δs becoming the infinitesimal change in distance, ds, and Δt dwindling to the correspondingly infinitesimal change in time, dt, so that

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[8] μένος: might, force, prowess; hence, in the case of a horse, *speed*.

[9] Keats, *Ode on a Grecian Urn*, line 17. Notice that even the rhythmic progression of the iambic line is halted as the pair "nev´-er, nev´-er" shift to trochees.

$$v = \frac{ds}{dt}.$$

In the idiom of differential calculus, this is the *first derivative* of distance with respect to time; and, inasmuch as the time-interval is infinitesimally small, it may be the closest we can come to expressing *instantaneous* speed. Notice, however, that this newly-discovered "speed" no longer belongs to the *moving being*, as Achilles' epithet did. It has, somehow, become sufficiently abstracted from the thing that moves to stand on its own as a *subject of discourse*.

{ Part II. Acceleration }

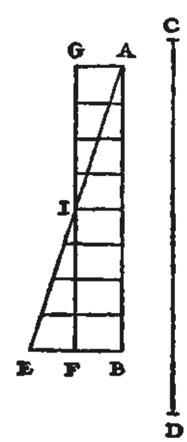
Meanwhile, in seventeenth century Venice, the three friends have proceeded from the topic of equable motion to that of *naturally accelerated* motion—the motion characteristic of freely falling bodies—which Galileo defines in this way:

> That motion is equably or uniformly accelerated which, abandoning rest, adds on to itself equal momenta of swiftness in equal times.[10]

This definition singles out the ability of accelerated motion to "add on to itself." But acceleration would seem to demand a very different kind of "adding" from what Galileo presupposed earlier when taking equimultiples. Acceleration is *intensification*; it is the intransitive increase of a general characteristic, "speed," while the other was union or amalgamation of two (or more) separate "speeds," as though speeds could be picked up and set down next to one another—the very way, indeed, in which the diagram for Proposition II represented them.

Do we find intensification of speed in the Homeric world? Since it represents the last-minute, come-from-behind burst that makes for a dramatic chariot race, we might expect to find examples among the funeral games in Book 23. But races in the *Iliad* seem not to be won by quick action, so much as they are lost by outside interference, as when Athene trips up Aias in the foot race (23.773–775).

Intransitive *acceleration*, then, seems not to be a salient characteristic of the Homeric world, and to that extent it is no part of that primordial notion of speed from which Galileo began. And perhaps it is for that reason that Galileo treats acceleration as though it were no different from that transitive mode of additivity which makes multiplication possible—the mode exhibited by both Antilochos and Zeus.



For in Proposition I of Natural Acceleration Galileo argues, first, that the parallels to BE represent "degrees of speed," while, second, "the aggregate" of the degrees included in quadrilateral AGFB are equal to "the aggregate" of the degrees included in triangle AEB. He is thus treating "degrees of speed" as though they were individual entities that may be combined and recombined just as literally as speeds themselves had been in Proposition II of Uniform Motion.

[10] NE 198. I have omitted the words "I say" to discourage misreading "that" as a relative pronoun; it is actually a demonstrative pronoun.

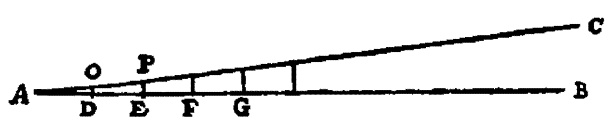
Here too, then, Galileo seems to be making explicit a notion barely expressed in the Iliad: that speed—at least in the human world—does not increase *of itself* but has to come *from* something. Why is this so? Because true *intensification* is hardly possible to conceive. How can something, *of itself*, become greater or less than it was? For that matter, how can a thing—*of itself*—change at all? In the *Iliad*, only the gods have the power to transform themselves; self-change is something divine, not natural. Galileo has, in effect, banished the divine.

In contrast to *intensification*, *addition* in the cumulative or arithmétic sense, is "change" in name only. One and one "make" two—but no entity, having been "one," turns into "two" by *addition*. In adding, each "one" remains what it is, just as, in Galileo's proposition, each of the "degrees of speed" that contribute to the speed of an accelerating body remains what it is. When "speeds" in this way become things in their own right, they instantly become *subjects of discourse*—"speed" becomes a grammatical subject. (Have you noticed those radar-activated road signs placed around the city? When one of them flashes: "Your speed is 35 miles per hour," speed is the subject, 35 is the predicate.)

I tried to show earlier that literal additivity of speeds leads almost inevitably to the representation of speed as a *quotient of numbers*; and quotients can be added. Then since Galileo has permitted himself to treat acceleration as a case of addition of speeds, he has in effect done what we have done; and if we therefore systematically replace his ratio language with algebraic language, we will only be doing openly what he did tacitly.

I will therefore not hesitate to construct a more analytical treatment of acceleration than Galileo does. To that end, consider Galileo's next proposition on Natural Acceleration—Proposition II, which demonstrates that

> If a moveable descends from rest in uniformly accelerated motion, the spaces run through in any times whatever are to each other as the duplicate ratio of the times; that is, as the squares of the times.[11]



I have rotated a portion of Galileo's diagram into what is, for us, the more familiar orientation of a graph of speed vs. time. Times are plotted along horizontal line AB; while the now-vertical lines represent the *speeds* which obtain at times D, E, F and G; each of them is thus an *instantaneous* speed—the kind of speed that would have been brought to grief by Zeno, had we not learned to understand it as a *first derivative*, represented

$$v = \frac{ds}{dt}.$$

And since the speed is changing over the whole course of the motion, it must be changing *at every moment* of the motion! The same pesky Zeno-itious difficulties arise as before; and we meet them in the same way, by considering

[11] Third Day, Naturally Accelerated Motion, NE 209, Stillman Drake translation.

$$\frac{dv}{dt},$$

the quotient of an *infinitesimal* change in speed, dv, by the correspondingly *infinitesimal* change in time, dt. In accordance with the definition of uniform acceleration, the quotient is *constant*.

Let us call the constant a, for we may regard it as a measure of the *acceleration*. In the idiom of differential calculus, acceleration is the *first derivative* of speed with respect to time. But we also said that speed itself was the derivative of distance with respect to time. Acceleration, then, is the *derivative of a derivative*. For

$$a = \frac{dv}{dt}$$

while

$$v = \frac{ds}{dt}\ ;$$

substituting, then, we obtain

$$a = \frac{dv}{dt} = \frac{d(v)}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt}.$$

Acceleration is then the time-derivative of the time-derivative of distance, or, in the idiom of differential calculus, the *second derivative* of distance with respect to time. I should point out, though, that the expression on the right above is today routinely written in this form:

$$\frac{d}{dt}\left(\frac{ds}{dt}\right).$$

Now, it seems that mathematicians must be inordinately fond of puns; for somebody—was it Leibniz?—evidently decided that since the expression above looks so much like a product, we might abbreviate it as though it *were* a product, and so write:

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) \rightarrow \frac{dds}{dtdt} \rightarrow \frac{d^2s}{dt^2}$$

You see how the last expression *is* just a pun, for it is not really a product of d's and t's. Yet instead of being greeted with hisses and boos, that notation actually remains predominant today. Like all puns, it is misleading if you fail to realize that it *is* a pun—for example, by calling it "d-squared-s over d-t-squared," and so embarrassing yourself in erudite company.

Fortunately there are alternative notations, which we are free to employ. There is Newton's economical practice of indicating derivatives with respect to *time* by *dots*; thus $\dot{s}$ and $\ddot{s}$ represent the first and second time-derivatives of s. Derivatives with respect to

Newton	Leibniz	Lagrange
$\dot{s}, \ddot{s}$	$\frac{ds}{dt}, \frac{d^2s}{d^2t}$	$s'(t)$, $s''(t)$ if $s = s(t)$
u', u''	$\frac{du}{dx}, \frac{du^2}{d^2x}$	$u'(x)$, $u''(x)$ if $u = u(x)$

distance he indicates with *primes*; so that u' and u'' represent the first and second distance-derivatives of u. Perhaps the most elegant notation of all is Lagrange's: if $s(t)$ is a function of t, $s'(t)$ represents its first derivative with respect to t, $s''(t)$ its second derivative with respect to t.

Likewise, if $u(x)$ is a function of x, $u'(x)$ represents its first derivative with respect to x, and so on.

Lagrange's notation is especially consequential, for it directs our attention to *functions* rather than to quantities. As he puts it,

> [The] object is not to find particular values of the quantities that are sought, but the system of operations to be performed on the given quantities.... The tableau of these operations represented by algebraic characters is what in algebra is called a *formula*, and when one quantity depends on other quantities, in such a way that it can be expressed by a formula which contains these quantities, we say then that it is a *function* of these same quantities.[12]

A focus on quantity finds its natural expression in a subject-predicate statement like "*y equals* 125." In contrast, the statement of a function emphasizes the *mathematical form* of a relationship—as, for example, in the function

$$y(x) = 5x^2.$$

To be sure, we can use this relationship to calculate a value of y if a value of x is given; but what the expression primarily conveys is the *structure* of the relationship, namely, that y is proportional to the *square* of x. As many of you will recognize, this relationship is a definitive characteristic of the parabola, and therefore expresses of something of the *being* of the parabola. To that extent, it has a closer kinship with the Homeric epithet than it does with the numerous subject-predicate statements—equations—that can be derived from it.

{ Part III. The Equation of a Plummeting Body }

At this point, having given what I hope was some useful attention to a few fundamentals, let us feel free to make unfettered use of the theorems of differential and integral calculus. And first, I should like us to derive the equivalent of Galileo's Proposition by means of those theorems. That will constitute the more analytical treatment I promised earlier. But we will still keep as close to Galileo's natural world as we can: we will not, for example, appeal to Newtonian gravity. Even though we refine Galileo's mathematics, we will retain his physics.

Let us therefore take up Galileo's case of a body falling from rest. If the acceleration is constant, we write this equation,

$$\frac{d^2s}{dt^2} = a\ ,$$

where a is constant. Integrating both sides with respect to time, we have:

$$\frac{ds}{dt} = \int a\,dt = at + C\ . \qquad (1)$$

Two remarks here. On the lefthand side, the integral of a *second* derivative is a *first* derivative. On the righthand side, remember that the result of any indefinite integration is always subject to an arbitrary additive constant C. To understand the meaning of that constant in the present case, note that if $t = 0$,

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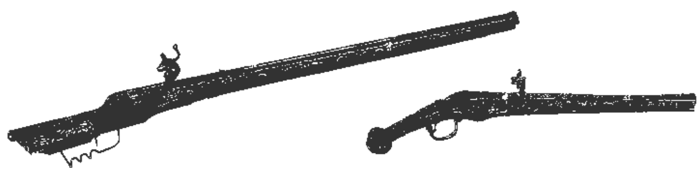
[12] *Résolution des équations numériques* (1798), Extract translated by Craig G. Fraser.

$$\frac{ds}{dt} = C.$$

Thus C will be the body's speed when the time is *zero*; that is, at the beginning of the motion. In Galileo's proposition, the body falls from *rest*, hence the initial speed is zero; so if *Galileo* were doing this calculation, he would set C equal to 0 in equation (1). But I would like us to consider the more inclusive case of a body that is not simply released from rest but is *thrust downward*. Galileo actually considers such a case on the Fourth Day:

> From a height of one hundred braccia or more, fire a lead bullet from an arquebus [*archibuso*] vertically downward on a stone pavement . . .[13]



The slide illustrates two 17th-century examples of the arquebus. This experiment is relegated to the Fourth Day because it is one of *violent* motion: the bullet is not simply released from rest (which would be a case of *natural acceleration*) but is *propelled* by an external agent—in this case by exploding gunpowder.

Judging from the friends' far-ranging discussion on the Third Day,[14] it is not easy to conceive beginning "from rest" as distinguished from beginning "with an initial speed." For help in this effort, let us call upon Milton. In Paradise Lost we read of Satan's expulsion from heaven in lines that are scarcely less spectacular than is the blast of Galileo's arquebus:

> Him the Almighty Power
> Hurl'd headlong flaming from th' ethereal sky,
> With hideous ruin and combustion, down
> To bottomless perdition, there to dwell . . .

When you read these lines, do you imagine the Almighty Power dangling Satan like a teabag, and then just letting him drop? Or do you rather see that blasted luminary as *launched*, *heaved*, *thrust*, and *driven* at the very first moment of his descent? Let us try to gain a more immediate sense of such a *launching*.

Here is a standard glass marble; I first hold it steady between my fingers, then I let it fall. . . This is *release from rest*, and it is obviously a most inadequate representation of Satan's experience. Now I take up an identical marble, but this time I will drive it down using this special mallet; its pin has length equal to the width of my fingers, so it cannot extend beyond them. Thus when I strike, the captured marble will be thrust from my fingers with whatever speed the mallet has attained—that will be the marble's initial speed. . .

I hope this exercise furnished a livelier sense of what is meant by "initial speed." Then, newly fortified with this example, recall the equation

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[13] Fourth Day, NE 279, Stillman Drake translation.

[14] Third Day, NE 199–200. I discussed this conversation in a 1986 lecture titled "All Beginnings Are Slow."

$$\frac{ds}{dt} = at + C \, ,$$

which we labeled (1) a little earlier, and where the constant C was to be given the value of the initial speed. If that value is v_0, then the equation becomes

$$\frac{ds}{dt} = at + v_0 \, .$$

We may then integrate both sides to obtain

$$s = \int (at + v_0)\, dt = \int at\, dt + \int v_0\, dt \;=\; \frac{1}{2}at^2 + v_0 t + C \, .$$

Again, on the leftmost side, the integral of the derivative of *s* is *s* itself. On the right, I once again use C to represent an arbitrary constant. And it is clear that when *t* is 0, *s* will be equal to C. Thus C is now the position from which the body begins its movement; it is the *initial position* of the body. If we represent that initial position by s_0, then the equation will become:

$$s = \frac{1}{2}at^2 + v_0 t + s_0 \, .$$

This, then, is the equation of motion for a body that is impelled from position s_0 with an initial speed v_0, and subject to a constant acceleration *a*.

Now why did I ask you to go through the derivation of the equation in red, which many of you no doubt derived last semester in Junior Lab? Partly it was to refresh the memory of that very derivation. But I also had something else in mind; and to that end, let me remind you of the interesting formalism of Lagrange and what it implies.

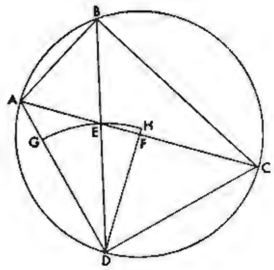
I pointed out earlier that Lagrange's notation focuses on *functions* rather than quantities. From that point of view, we did not derive "an equation" for descending bodies; rather, we found a *function*: specifically, we found *s* as a function of *t*, having the form:

$$s = \frac{1}{2}at^2 + v_0 t + s_0 \, .$$

But our search for that function could not even have begun had we not presupposed, with Galileo, that acceleration—that is, the second derivative—was constant. We had to start from a *known* derivative in order to begin integrating and so obtain the function that we did. Moreover, we had to assume that we were beginning with the *highest* derivative. If that is not clear at this point, it should become evident very soon.

But how can we specify a derivative if we cannot even express the function? What if we don't know which derivative is the highest? What if the function cannot be reduced to algebraic terms at all? That, in a way, was Ptolemy's problem when he set out to compute his Table of Chords.[15]

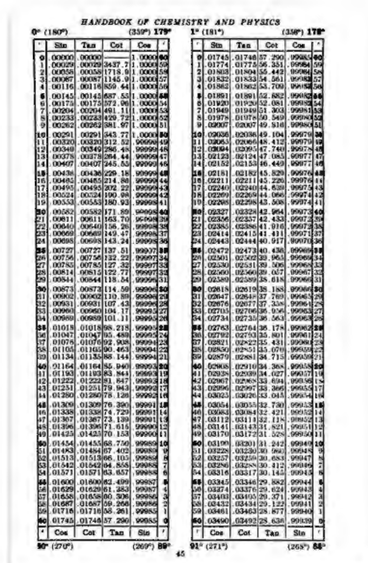
[15] Ptolemy, *Almagest*, Book I, Chapters 10–11.



II. TABLE OF CHORDS

Arcs	Chords			Sixtieths				Arcs	Chords			Sixtieths			
½	0	31	25	0	1	2	50	16½	17	13	9	0	1	2	10
1	1	2	50	0	1	2	50	17	17	44	14	0	1	2	7
1½	1	34	15	0	1	2	50	17½	18	15	17	0	1	2	5
2	2	5	40	0	1	2	50	18	18	46	19	0	1	2	2
2½	2	37	4	0	1	2	48	18½	19	17	21	0	1	2	0
3	3	8	28	0	1	2	48	19	19	48	21	0	1	1	57
3½	3	39	52	0	1	2	48	19½	20	19	19	0	1	1	54
4	4	11	16	0	1	2	47	20	20	50	16	0	1	1	51
4½	4	42	40	0	1	2	47	20½	21	21	11	0	1	1	48
5	5	14	4	0	1	2	46	21	21	52	6	0	1	1	45
5½	5	45	27	0	1	2	45	21½	22	22	58	0	1	1	42
6	6	16	49	0	1	2	44	22	22	53	49	0	1	1	39
6½	6	48	11	0	1	2	43	22½	23	24	39	0	1	1	36
7	7	19	33	0	1	2	42	23	23	55	27	0	1	1	33
7½	7	50	54	0	1	2	41	23½	24	26	13	0	1	1	30
8	8	22	15	0	1	2	40	24	24	56	58	0	1	1	26
8½	8	53	35	0	1	2	39	24½	25	27	41	0	1	1	22
9	9	24	54	0	1	2	38	25	25	58	22	0	1	1	19
9½	9	56	13	0	1	2	37	25½	26	29	1	0	1	1	15
10	10	27	32	0	1	2	35	26	26	59	38	0	1	1	11

There exists no arithmetical method for exact computation of the chord of a given arc (what we now call twice the sine of half the given angle). Ptolemy did develop a geometric method; but for angles of one degree or less it was accurate only to about 20 "thirds" in the sexigesimal system—about four decimal places.[16] But long before the 1920s, physicists and engineers were making routine use of trigonometric tables like this one, accurate to *five* decimal places.[17] Where did that extra place come from? And what about today's pocket calculators, which instantly figure values to as many as *nine* decimal places? If the *sine* cannot be reduced to an algebraic formula, what in the world is the calculator calculating?


HANDBOOK OF CHEMISTRY AND PHYSICS

0° (180°) (359°) 179° 1° (181°) (358°) 178°

Sin Tan Cot Cos

90° (270°) (269°) 89° 91° (271°) (268°) 88°

In asking this, I am not merely marveling at the digital calculator's speed. Even if you were the Speedy Achilles of Calculating, without a formula to tell you what to add, subtract, multiply, or divide, you could not even make a beginning; you would be as paralyzed as was Zeno's Achilles in his bid to overtake the tortoise. "No algebra, no computation," as one of my teachers used to say. Or so it would seem.

For there is a way to calculate a value, even if we have no algebraic expression for the function. Instead of beginning with the *highest* derivative we begin with the *lowest*, that is to say, the *zeroth* derivative: a particular value of the function itself without taking any derivatives at all. Although that procedure may seem opaque at first, it results in what we now call the Taylor series—a train of reasoning which enables us to build up an algebraic expression, term by term, for functions like the sine—functions which, strictly speaking, have no algebraic expression.

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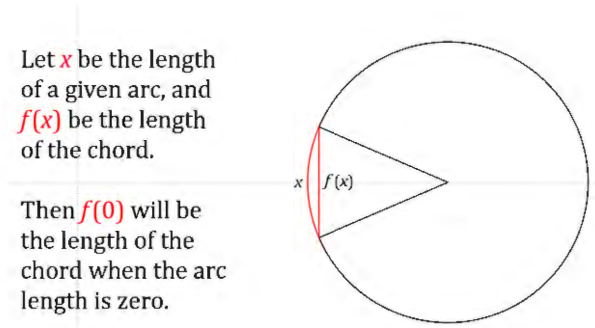
[16] *Almagest* Book I, Chapter 10, H46.

[17] *Handbook of Chemistry and Physics*, 13th ed., The Chemical Rubber Company (1928).

{ Part IV. The Taylor Series }

The Taylor series was formulated in 1715 by Brook Taylor[18]—the same Brook Taylor juniors know in connection with vibrating strings. I will confine my discussion to a special case of that series, called the Maclaurin series; but you will find a résumé of the general series in the handouts here on the table, which you are welcome to take after the talk.

Imagine, then, that you are Ptolemy in search of the chord of some arc *x*; or, [SLIDE 61] otherwise expressed, you are in search of some function $f(x)$, the "chord-function." And suppose, further, that you know the value of that function when the arc $x = 0$.[19] In fact, you



do know the chord of an arc having zero length—it is, obviously, *zero*! But to keep our thinking general, let us call that value $f(0)$. Can we possibly determine the chord of some other arc x, even though we don't know the function explicitly? Taylor's method undertakes, in effect, a systematic tabulation of *all the derivatives* which the unknown function could possibly have.

In anticipation of that, let us prove a little lemma: *I say that if any derivative of a function is constant* (but not zero)*, that derivative is the highest derivative*.

For suppose that, say, the third derivative of some function is constant, but not zero. Then it is clear that the fourth derivative will be *zero*, since the derivative of a constant is zero. Then clearly the fifth, sixth, and *all higher derivatives* will also be zero—in other words, the derivative that equals a nonzero constant is the highest derivative. In a similar way we can prove the converse: that if a function has a highest derivative, that derivative is a nonzero constant.

In the following discussion, I will go through the math pretty quickly; but it is written out in full in the handout. First, then, suppose the chord-function has *no derivatives* (that is, that each of its derivatives is equal to zero), then the "function" will actually be a *constant*, and it will have the same value at any x as it does at 0. We write:

$$f(x) = f(0) = A\,,$$

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[18] Brook Taylor. *Methodus Incrementorum Directa & Inversa* (1715). Annotated translation by Ian Bruce at https:www.17centurymaths.comcontentstaylorscontents.html. James Gregory stated a substantially identical relation in 1671, but only for certain specified functions. St. John's students may enjoy knowing that, like them, Taylor too attended St. John's College—of course, his was St. John's College, Cambridge.

[19] It is this singling out the region of *x* close to zero that constitutes the Maclaurin series. The actual Taylor series applies to a small region about *any x*.

and for simplicity, I represent that value by A. If such *were* the case, then as the arc grew from 0 to x, the length of the chord would remain equal to A—and the result would be

$$f(x) = A\,.$$

Obviously this cannot be true in the case that is now before us: the chord cannot possibly be constant if the arc increases. So it is clear that the chord-function must have, *at least*, a first derivative.

If, then, the chord-function has a *first* derivative, and if that derivative is highest, it will also be constant; the first derivative evaluated at *any* x will equal its value when $x = 0$. And if we call that value B, we shall have

$$f'(x) = f'(0) = B;$$

so that, integrating

$$f(x) = \int B\,dx = B\int dx = Bx\,.$$

What this means is that when the arc grows from 0 to x, the existence of a first derivative will contribute Bx to the value which the chord had to begin with, for a total length of

$$f(x) = A + Bx\,.$$

But perhaps the chord-function has a *second* derivative, and that derivative is the highest. Then it will be constant, or

$$f''(x) = f''(0) = C.$$

Integrating twice, then,

$$f'(x) = \int C\,dx = C\int dx = Cx$$

$$f(x) = \int Cx\,dx = C\int x\,dx = \frac{1}{2}Cx^2\,.$$

Again, this means that when the arc grows from 0 to x, the existence of a second derivative will contribute $\frac{1}{2}Cx^2$ to the value which the chord had already, for a total length of

$$f(x) = A + Bx + \frac{1}{2}Cx^2\,.$$

Now, let's stop here for a moment and notice the remarkable similarity between this function and our earlier equation of motion if we merely reverse the order of terms—as you see here:

$$s = s_0 + v_0t + \frac{1}{2}at^2\,;$$

for s_0 in the equation corresponds to capital A in the function; and v_0 in the equation corresponds to B in the function. Finally, a in the equation corresponds to C in the function. Thus we might have derived the Galilean equation of motion backwards, instead of the way we actually did. Had we done so, the question would not have been, as it was for us, one of knowing which derivative to *start* with, but rather knowing at which derivative we may *stop*. "Where to stop" is the question in our present endeavor, the quest for a function that will express the chord of arc x. Unlike the case of natural acceleration, we have no reason to believe that the derivatives of the chord-function stop at the second. There may be a *third* derivative; and if *that* is the highest derivative it will be constant; we will have

$$f'''(x) = f'''(0) = D;$$

and thus, integrating three times, we obtain

$$f''(x) = \int D\,dx = D \int dx = Dx$$
$$f'(x) = \int Dx\,dx = D \int x\,dx = \frac{1}{2}Dx^2$$
$$f(x) = \int \frac{1}{2}Dx^2\,dx = \frac{1}{2}D \int x^2\,dx = \frac{1}{2}\cdot\frac{1}{3}Dx^3\ .$$

As before, this is the quantity which the existence of a third derivative contributes to the value of the function; and when we add this quantity to the value it already had, we obtain the result shown here:

$$f(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{2}\cdot\frac{1}{3}Dx^3\ .$$

Clearly, there is no end to this process so long as there is the possibility of another derivative. But even if the chord-function has *infinitely many derivatives* (that is, if it is a *transcendental* function—which, by the way, it is) we will come closer and closer to its complete expression with each newly added term.[20] You can see the developing pattern more easily if we replace *A*, *B*, *C* and *D* with their Lagrangian expressions, to obtain this equation:

$$f(x) = f(0) + f'(0)\cdot x\ + f''(0)\cdot\frac{1}{2}x^2 + f'''(0)\cdot\frac{1}{2}\cdot\frac{1}{3}x^3\ ,$$

or, written more compactly,

$$f(x) = f(0) + \frac{f'(0)}{1}x\ + \frac{f''(0)}{1\cdot 2}x^2 + \frac{f'''(0)}{1\cdot 2\cdot 3}x^3\ \cdots$$

You see the regularity and beauty of this pattern: with each term we have a higher level of differentiation, a higher power of *x*, and a higher factor in the denominator. With this extremely powerful method we can calculate a Ptolemaic Table of Chords—or a modern sine table—to any degree of precision we desire. The more terms we include in the series, the more nearly accurate will our calculations be. We can thus endeavor, at least, to express *any* continuously differentiable function, no matter what its form. In a very real sense, we can say that *form* has superseded *magnitude*.

You have been kind enough to join me in an exploration that proceeded from Troy to Venice; and I hope you feel that Galileo's Venice makes a little more sense when we can discover in it traces of Homer's Troy. But since we ended up in Taylor's London, it is appropriate to ask, "Is there an abiding feature, a thread of Ariadne, that binds all three of these settings?"

If there is one, I think it centers on the forms of *discourse*. When Venice commandeers Achilles he loses an ontological attribute; for s*peed*, no longer an epithet but a magnitude, becomes only a conditional predicate. Further, as a measurable magnitude, "speed" replaces the moving being as the grammatical subject—with the *measure* as its predicate.

This transformation of subject and predicate culminated with Galileo. But we found that when Galileo's mathematical method was reversed in direction, it carried us to still another stage, one in which the subject/predicate construction altogether *disappeared*. For that

[20] The larger the value of *x*, of course, the more terms we will need to meet a given standard of closeness.

reversal led to the Taylor series, which, as it amasses more and more terms, passes beyond merely calculating *the value of a quantity* to reveal *the form of a function*. There is no subject; there is no predicate; there is only *form*, which, I suggested earlier, enjoys a certain kinship with Homeric epithet as expressive of a mode of being rather than a mode of speaking. If indeed we were heartened to find a trace of Homeric narrative in Galileo's geometrical mathematics, may I hope that an echo of Homer's voice might find a similar welcome when it appears in analytical calculus?