Lecture 1: Meet Your New Best Friend: the Square Root of Negative One

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Introduction

This is the first of a series of six lectures on algebra. After today's lecture, there will be four more talks on successive Tuesday evenings, each unpacking a step along the way to the final result that I want to share with you, Gauss's demonstration of the Constructibility of the Seventeen-Sided Polygon or "Heptadecagon." (The schedule is available on the handouts.) Afterwards, on October 16, there will be a final Wednesday afternoon lecture that very briefly recaps the contents of the Tuesday evening technical talks and concludes with some reflections on the significance of the square root of negative one to the foundations of arithmetic and the relation of ordinary experience to mathematics.

This series of talks grew out of a remark that a former dean of the College made to me years ago. This person, whom I respect and admire greatly, said something I thought was seriously questionable. "Algebra is boring," he opined. "Our students love geometry," he said, "because its beauty strikes them immediately. But algebra is just a tool, a technique. Nobody wants to spend any class time studying it. It's just dull."

I took those words as a challenge. It didn't seem plausible that sane people would devote countless hours of intense intellectual effort to something that is inherently dull, at least without being paid a *lot* of money. (Algebraists generally are *not* highly paid.) Do they really enjoy tedium?

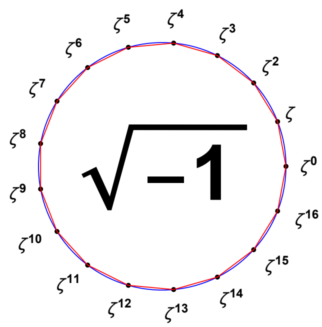
No, they don't. Algebraic structures have a real beauty, even if it takes a little work to notice and appreciate it.

For better or for worse, algebraic beauty is invisible beauty, and the taste for it is an acquired taste, like that for single malt scotch, twelve-tone music or the word-play of *Finnegans Wake*. My on-again, off-again quest, therefore, for many years has therefore been to find the some entryway into what algebra has to offer, something more appealing than exercises in factoring polynomials. I wanted to find the algebraic equivalent of the Pythagorean Theorem, some result that would make someone stand still in wonder and say, "Whoa! Really?," as Thomas Hobbes reportedly did when he saw a copy of Euclid open to proposition 47 of book one. Legend has it that he stood transfixed at a library table for hours reading the entire first book of the *Elements* BACKWARD until he arrived at the postulates. That encounter reportedly made him "in love with geometrie." (John Aubrey, *Brief Lives* (c. 1700))

What, then, would make someone "in love with algebra?" If anything could do it, I think it would be Carl F. Gauss's construction of the heptadecagon. It's a beautiful result, and the pathway to it, while not perfectly smooth, requires only a few hours of preliminary work, not years. Also, for this audience, it speaks directly to geometrical demonstrations that all St. John's students have encountered by the middle of the first semester of their freshman year, namely, the propositions of Book Four of Euclid's *Elements* where we see the construction of regular-sided polygons in circles. Euclid shows how to construct the equilateral triangle, the square, the pentagon -- that's a hard one! -- and the hexagon inside a given circle. Then, without explanation, he skips ahead to the fifteen-gon (Elements IV, 16). Then he stops. Why? Euclid, characteristically laconic, says nothing.

Two thousand years later, a very young Carl Friedrich Gauss provided the answer.

However, in order reach that result, you need to come to terms with [horror suspense sound effect] *the square root of negative one.* [mad scientist laugh sound effect]



Some people have difficulty accepting this number. They say, for instance, that they are put off by the fact that it doesn't exist. Which is ridiculous! You shouldn't let so trivial a problem prevent you from embracing this concept, for the square root of negative one, also denoted by the single letter *i,* is the gateway to an algebraic realm of amazing results. Through it we come to the Complex Number Field. This is a spectacular realm where we see hidden machinery that links algebra and trigonometry, wonderfully expressed in DeMoivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\,\theta;$$

where we find Euler's famous identity "$e^{i\pi} = -1$," where all polynomial equations can be completely decomposed into linear factors and where we hear

> the buzzin' of the bees in the cigarette trees
> 'round the soda water fountains!

No, wait. That's the Big Rock Candy Mountains. Never mind. The Complex Number Field is still an algebraic paradise. It is here that you must go to find the result that I most want to show you, the construction of the heptadecagon.

Gauss's result is not overly complicated, but it does involve several separate stages, so I thought it best not to try to jam all of it into a single talk. Doing that would generate more confusion than understanding. So, all I want to do today is to set up the issue of the square root of negative one by describing how it first appeared, how it was at first dismissed, and how an algebraic triumph led an Italian algebraist to reconsider the possibility that it might not be gibberish but an important numerical results.

Start Simple: The Linear Equation

We will be concerned with polynomial formulas and their solutions. The simplest polynomial formula is an equation in the first degree, that is, one where the variable -- we'll call it *x* -- appears only to the first power:

$$x - a = 0,$$

Here "x" is an unknown quantity and "a" is something given. To be more particular, we might have:

$$x - 3 = 0.$$

What is x? In this case, x is 3. Is that obvious? You don't need to do the explicit manipulation of adding 3 to both sides of the equation, although you could:

$$x - 3 + 3 = 0 + 3$$

$$x = 3.$$

Already there are, in fact, subtleties and complexities that could be discussed at length. What are these quantities "x" and "a"? Are they lengths? areas? numbers? magnitudes? Do we know? Do we care? Are they *particular* lengths or numbers? It is possible to imagine a "general" quantity that is nothing in particular? Even if we cannot *imagine* it, can we *conceive* of it? What is a "variable" like *x*? How is a variable similar to or different from a "constant term" like *a?* In a situation in which we don't know what "x" and "a" are, are there differences in the *manner* in which we "don't know *x*" and in which we "don't know *a?*" Can we "add the same thing to both sides of an equation" if we have no idea what those things are?

All these questions are interesting and important. However, I'm going to pass by all of them because the real concern of this lecture lies further down the road in the direction of more complex equations.

The Next Step: the quadratic, $x^2 + b x + c = 0$

I hope you found the linear equation x - 3 = 0 easy to solve. Things get more complicated quickly.

In the realm of polynomials, the next step in complexity takes us to the *quadratic* equation:

$$x^2 + b x + c = 0$$

where “b” and “c” are rational numbers. (Why start with “b” and not “a”? Because the really simple form is “$a x^2 + b x + c = 0$” where “a,” “b,” and “c” are all integers. But it is convenient to divide out the lead term “a” and re-define *b* and *c* so that one has a polynomial whose leading coefficient is 1. Such an expression is called a “monic polynomial.”)

There is a path to solving this equation quickly and reliably. It’s called “completing the square.” I suspect it may be familiar to many of you. In case it isn’t, I will review it here. Stepping through the derivation of the quadratic formula will be useful when we turn in a moment to the next step, the cubic.

Consider: what does a “perfect square” polynomial look like in algebra? That is, what do we get when we multiply a single factor by itself. Try making one:

$$(x - b)(x - b) = x^2 - 2 b x + b^2$$

That’s what a “perfect square” polynomial looks like, one whose two solutions are both *b.* If someone posed that polynomial for us to solve,

$$x^2 - 2 b x + b^2 = 0$$

life would be easy! We would just take the square root of both sides:

$$\sqrt{x^2 - 2 b x + b^2} = \sqrt{0} = 0$$

$$\sqrt{(x - b)^2} = 0$$

$$x - b = 0$$

$$x = b.$$

Sadly, $x^2 + b x + c = 0$ is *not* perfect square. Happily, however, we can *make it into a perfect square,* or at least get close enough. Look again at the general equation for a perfect square:

$$x^2 - 2 b x + b^2 = 0$$

Notice the relation between the coefficient of x (- 2 b) and the constant term (b^2). If you take the coefficient of x, divide by 2 then square it, you get the constant term. Now look at the equation we actually have:

$$x^2 + b x + c = 0$$

By brute force, let's *make* a perfect square. First, subtract c from both sides of the equation, just to "clear the decks."

$$x^2 + b\,x = -c$$

Take half the middle term ($\frac{b}{2}$) and square it: you get $\frac{b^2}{4}$. Add that to both sides of our equation:

$$x^2 + b\,x + \frac{b^2}{4} = \frac{b^2}{4} - c = \frac{b^2 - 4\,c}{4}$$

Now, on the left, you have a *perfect square:* $x^2 + b\,x + \frac{b^2}{4} = \left(x - \frac{b}{2}\right)^2$. This is great! Take the square root of both sides:

$$\sqrt{x^2 + b\,x + \frac{b^2}{4}} = \sqrt{\frac{b^2 - 4\,c}{4}}$$

On the left, we have the square root of a perfect square, a situation that we deliberately contrived:

$$\sqrt{\left(x + \frac{b}{2}\right)^2} = \sqrt{\frac{b^2 - 4\,c}{4}}$$

$\pm\left(x + \frac{b}{2}\right) = \frac{\sqrt{b^2 - 4\,c}}{2}$ (We need to say "±" because both $\left(x + \frac{b}{2}\right)$ and $-\left(x + \frac{b}{2}\right)$ when squared give the same result.)

$$x = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4\,c}}{2}$$

This is the standard form of the quadratic formula for a monic, quadratic polynomial. Plug in the coefficients *b* and *c,* turn the crank and out pop two values for *x.*

A Cute Quadratic Trick

Before we go on, however, there's a clever little trick involving quadratics that I want to show you. It will be used many times in what follows.

Suppose you have two *different* factors, (x - a) and (x - b) of a polynomial equation.

$$(x - a)\,(x - b) = 0$$

Multiply and expand:

$$x^2 - (a + b)x + ab = 0$$

Look at the coefficient of x and at the constant term: a + b and a times b. The first is the sum of a and b, the second is the product, where a and b are the two roots of the polynomial. This observation can easily be generalized. If we had *three* terms:

$$(x - a)(x - b)(x - c) = 0$$

The expanded version would look like this:

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x + abc = 0$$

For the fourth degree:

$$(x - a)(x - b)(x - c)(x - d) = 0$$

The expanded version would look like this:

$$x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + bcd)x + abcd = 0$$

Maybe you see where this is going: the first non-zero coefficient is always the *sum of all the roots.* The second is the *sum of all the roots taken two at a time.* The third is *the sum of all the roots taken three at a time.* The constant term, when you get to it, is always the *product of all the roots.* These patterns are very interesting and very useful. The patterns that you see here can be described by saying that the coefficients are "symmetric functions" of the roots because, as you can see, each of *a, b, c* play the same role in each expression. Exploration of symmetric functions is fascinating, but for us right now it is beside the point.

Look back at the second degree equation, and particularly at the coefficients of the equation: the coefficient of x is *the sum of the solutions a and b.* The constant term, is *the product of the solutions a and b.* Pause over that for a second: just looking at the polynomial may not tell us the two solutions immediately, but even a glance at the coefficients give us the *sum* and the *product* of the solutions.

Now, turn that observation *inside out.* Suppose you have two unknown numbers, call them r_1 and r_2. Suppose further that you don't know what these two numbers are, but you DO know *their sum* and *their product.* In that case, you can make a quadratic equation that has these two numbers as its solutions:

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

The solutions of this equation can be found with the quadratic formula:

$$\frac{(r_1+r_2) \pm\sqrt{(r_1+r_2)^2 - 4 r_1 r_2}}{2} = x$$

You may say to yourself, “That’s great, Mr. Franks. But how often, really, does it happen that I come to know the *sum* and the *product* of two numbers without knowing what those numbers are individually?” Well, in your daily life, maybe not so often. However, in the algebraic journey that lies before us, this little trick is going to show up more often than you may imagine.

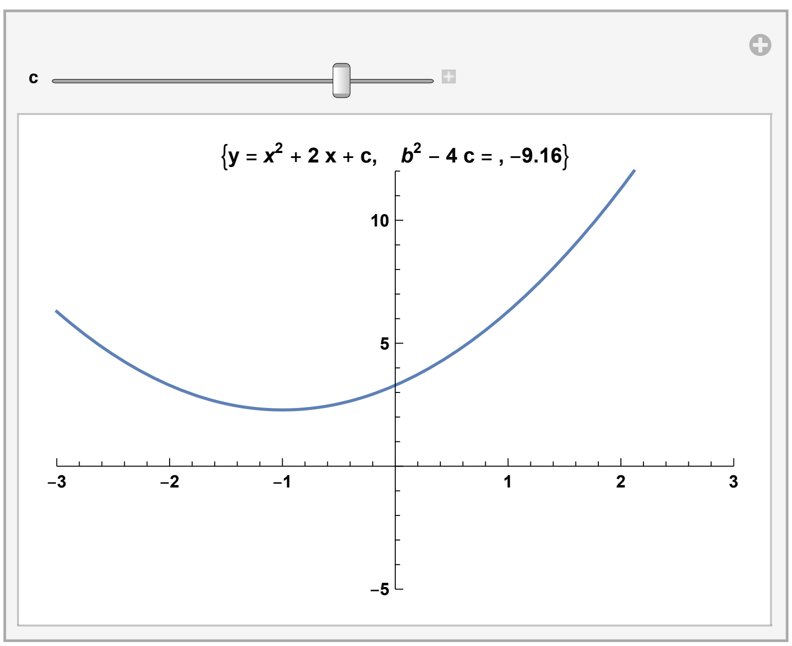
What Happens When Things Go Wrong

So far, we have constructed the Quadratic Formula, which solves a general quadratic equation:

$$x^2 + b x + c = 0 \qquad \Longrightarrow \qquad x = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4 c}}{2}$$

You notice that the solution contains a radical, $\sqrt{b^2 - 4 c}$. If the quantity under this radical, which is called “the *discriminant,”* is positive, all is well. But if that quantity becomes negative, that is, if 4c is greater than b^2, then we have a problem.

Here is a graph of the equation $y = x^2 + 2 x + c$. We have arranged matters so that we can vary the value of the constant term *c:*



The “solutions” -- that is, the x values where the function equals zero, are given by:

$$x = \frac{-2 \pm \sqrt{4 - 4 c}}{2} = \frac{-2 \pm 2\sqrt{1 - c}}{2} = -1 \pm \sqrt{1 - c}$$

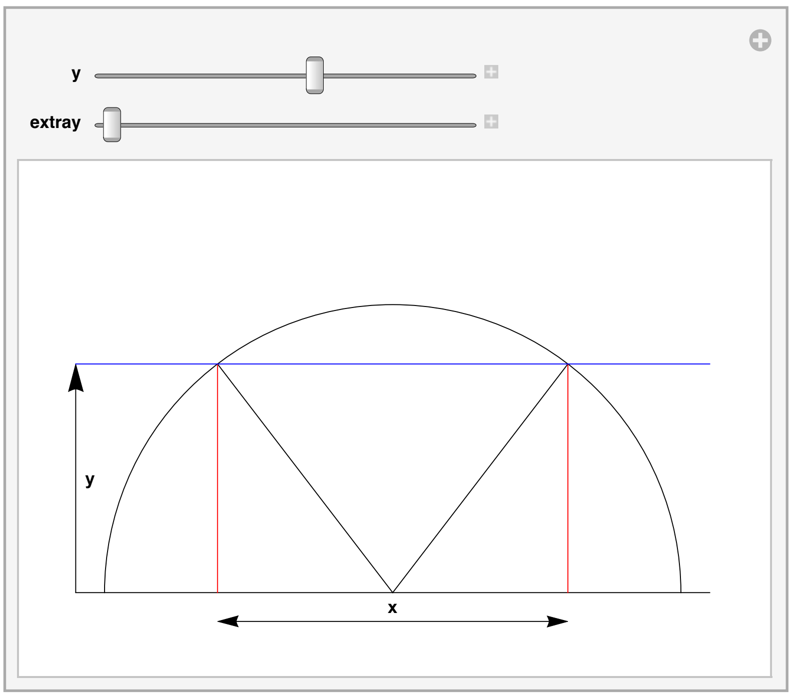
When c is less than one, the expression is positive and there are two solutions. When *c* is equal to 1, the expression under the radical is zero and there is one solution. When it is greater than one, the expression under the radical is less than one and, as you can see graphically, there appears to be *no* solution. From this example, we might conclude that the square root of a negative number means "impossible."

Another Example

Descartes came to this conclusion looking at a slightly different example. He considered the expression:

$$x = \sqrt{1 - y^2}$$

He could picture this by an illustration like the one below. Consider a semi-circle with radius 1. Draw a line parallel to the diameter with a variable height, y.



So long as y lies between zero and one, the quantity under the radical is positive and there are two solutions, a negative one and a positive one, indicated by the intersection of the horizontal blue line with the semicircle. When y reaches the value 1, then the horizontal line is tangent to the semicircle and there is one solution only. When y is *greater* than 1, the horizontal line misses the semicircle. It looks as if then there are no solutions at all.

Tentative conclusion: when a radical contains a negative sign, the formula is meaningless. There is *no* solution to the equation, and the expression should be rejected as absurd.

That conclusion was generally accepted before Rafael Bombelli began to work with the cubic equation.

The Cubic

The cubic equation poses greater challenges than the quadratic. The quadratic equation is not exactly *simple*, but its solution has been known for a long time. Babylonians had techniques that were more or less equivalent of solving a quadratic equation. Some of Euclid's geometrical manipulations in Book II of the *Elements* also answer questions that are closely analogous to finding the solution of the general quadratic equation.

By contrast, the solution for the cubic equation was not published until the sixteenth century. The formula for the cubic is known as "Cardano's Formula," named after Girolamo Cardano who succeeded where many others had failed … by stealing it from the man who invented it, Niccolo Tartaglia. Cardano published this formula in his [Cardano's] book, *The Great Art or the Rules of Algebra* (1545). Let this be a lesson to you: if you want to make a name for yourself in mathematics, steal freely and publish early. (This is an example of "Stigler's Law of Eponomy" which states that no result in science or mathematics is named after the person who first discovered it. See, Tom Lehrer's song, *Lobachevsky.*)

Because Cardano's formula is important, I propose to walk through its derivation here. I am aware, however, that it is hard to follow a sequence of algebraic steps in a lecture format; that's why I have printed copies for anyone who wishes to review the derivation later at her or his leisure. The lecture will also be posted on some part of the College's web page. Of course, you are also free (if you wish) just to accept the result on faith.

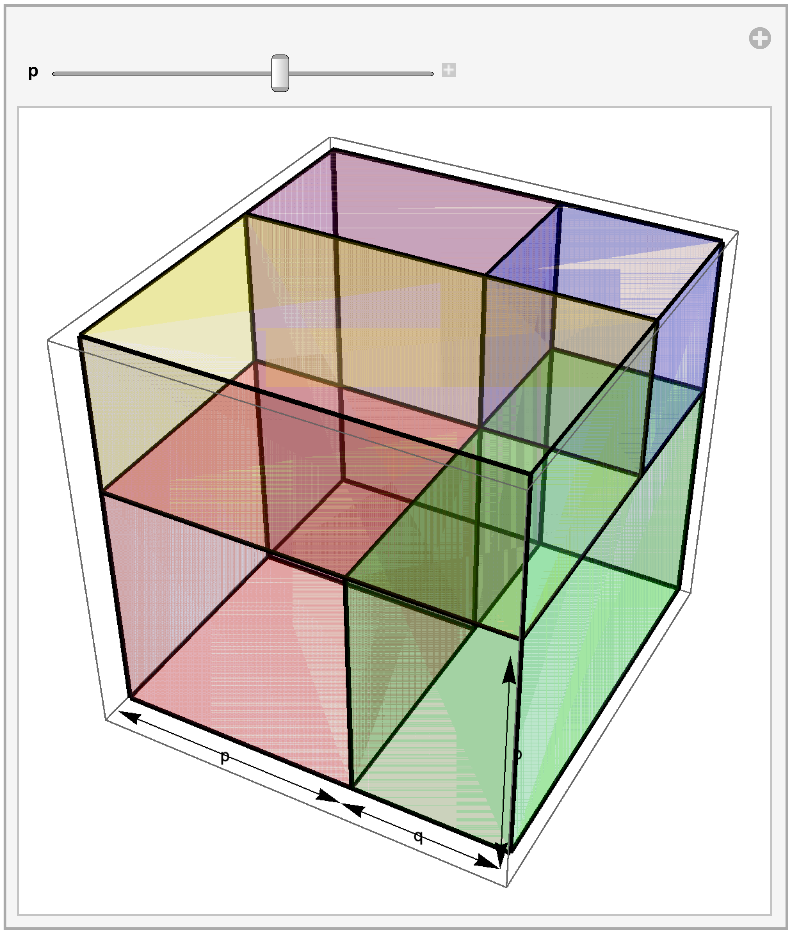
One begins, not with the full form of the cubic, but with the "depressed cubic," a formula that has been manipulated so that the x^2 term disappears. There is a routine procedure for making this happen, so it does not limit the generality of the demonstration. (The procedure for "depressing the cubic" is included as an appendix to the printed version of this lecture for anyone to examine at leisure.) Thus, we begin with:

$$x^3 + c\,x + d = 0$$

We want to know what x is. To solve this equation, Cardano -- or Tartaglia, really -- came up with a special trick: re-conceive x as divided into two parts, p and q so that

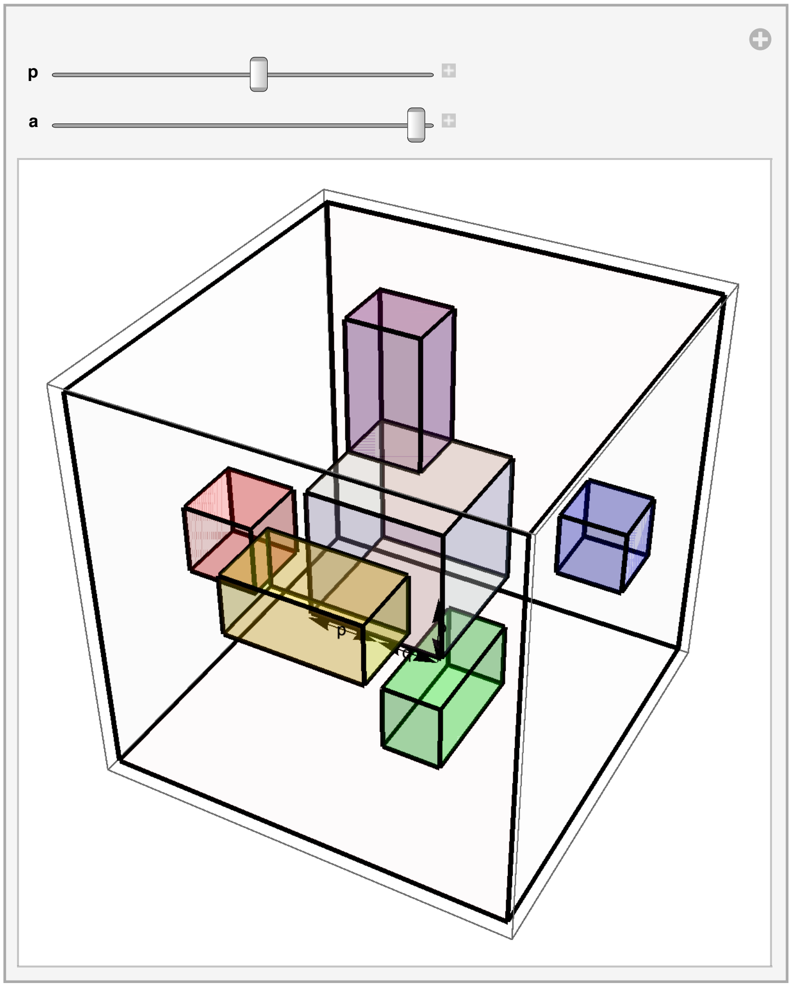
$$x = p + q.$$

Imagine, then, a cube whose whole side is x, a length that has been divided into two parts, p and q:



We can see that the cube is broken up into

(i) a cube of side p (red);
(ii) a cube of side q (blue); and
(iii) three “slabs” of volume *p q (p + q)* (yellow, green and purple).



Symbolically, we have:

$$(p+q)^3 = p^3 + 3p^2 q + 3pq^2 + q^3 = 3pq(p+q) + (p^3+q^3)$$

or

$$(p+q)^3 - 3pq(p+q) - p^3 - q^3 = 0$$

which gives us:

$$\underbrace{(p+q)^3}_{x^3} + \underbrace{(-3pq)}_{c}\underbrace{(p+q)}_{x} + \underbrace{(-p^3 - q^3)}_{d} = 0$$

Compare this to our depressed cubic:

$$x^3 + cx + d = 0$$

The two are the same *PROVIDED:*

$-3\,p\,q = c$ and $-(p^3 + q^3) = d$ or

$p\,q = \frac{c}{-3}$ and $-(p^3 + q^3) = d$ or, cubing the first expression --

$\mathbf{p^3\,q^3} = \frac{-c^3}{27}$ and $\mathbf{(p^3 + q^3)} = -d$

Now look: we have expressions for the the ***sum*** and the ***product*** of the two quantities, p^3 and q^3. Therefore, we can construct the quadratic equation. (I told you this procedure would be helpful!) I'll use "w" as a variable:

$w^2 - (p^3 + q^3)\,w + p^3\,q^3 = 0$ which will have p^3 and q^3 as it solutions. Substitute the values we just determined:

$$w^2 + d\,w - \left(\frac{c}{3}\right)^3 = 0$$

whose two solutions are:

$$w = \frac{-d \pm \sqrt{(d)^2 + 4\left(\frac{c}{3}\right)^3}}{2} = \frac{-d \pm \sqrt{4\left(\frac{d}{2}\right)^2 + 4\left(\frac{c}{3}\right)^3}}{2} = \frac{-d \pm 2\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}}{2} = \frac{-d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}$$

The two solutions of this are p^3 and q^3. If we take the cube root of each and add them, we get p + q = x:

$$x = \underbrace{\sqrt[3]{\frac{-d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}}}_{p} + \underbrace{\sqrt[3]{\frac{-d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}}}_{q}$$

That is Cardano's formula -- the one he stole from Tartaglia -- for solving the cubic equation.

What Happens When Things Go Wrong?

For the Quadratic: Apparent Impossibility

If you have followed so far, we have two formulas, one for second degree (quadratic) equations and one for third degree (cubic) equations.

The quadratic formula:

For the equation: $x^2 + b\,x + c = 0$

x is given by: $x = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4\,c}}{2}$

The cubic formula (“Cardano’s Formula”):

For the equation: $x^3 + c\,x + d = 0$

x is given by: $x = \sqrt[3]{\frac{-d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}}$

This is all very well … ***except*** if the coefficients are such that the quantities under the square-root signs become negative.

We have already seen what happens to the quadratic when the quantity under the radical is negative: the formula seems to give no answer at all and the expression appears to be meaningless.

A priori, we seem to have no reason to suspect that the cubic will act differently.

But it does.

You Can Use Cardano’s Formula to Solve Cubics

Sometimes, Cardano’s formula works just fine. Consider, just as an example:

$0 = x^3 + 6\,x + 20$ where $c = 6$ and $d = 20$

Apply the formula to get:

$$x = \sqrt[3]{\frac{-d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}}$$

$$x = \sqrt[3]{\frac{-20}{2} + \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3}} + \sqrt[3]{\frac{-20}{2} - \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3}}$$

$$x = \sqrt[3]{-10 + \sqrt{100 + (2)^3}} + \sqrt[3]{-10 - \sqrt{100 + (2)^3}}$$

$$x = \sqrt[3]{-10 + \sqrt{108}} + \sqrt[3]{-10 - \sqrt{108}}$$

$$x = \sqrt[3]{-10 + 12\sqrt{3}} + \sqrt[3]{-10 - 12\sqrt{3}}$$

$$x = 0.73205 + (-2.73205)$$

$$x = -2$$

... which really is one of the solutions, as you can check. (The other two are imaginary, which you can find after you have factored out (x + 2) from $0 = x^3 + 6x + 20$.

$$x^3 + 6x + 20 = (x + 2)(x^2 - 2x + 10)$$

You can solve $x^2 - 2x + 10 = 0$ with the quadratic formula.)

However, Sometimes Things are Very Weird

But things can also go terribly, terribly wrong. Take another example:

$$x^3 - 15x - 4 = 0$$

Here, c = - 15 and d = - 4. Put the values into Cardano’s Formula:

$$x = \sqrt[3]{\frac{-d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}}$$

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^3}} + \sqrt[3]{\frac{4}{2} - \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^3}}$$

$$x = \sqrt[3]{2 + \sqrt{4 + -125}} + \sqrt[3]{2 - \sqrt{4 + -125}}$$

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}.$$

You have an imaginary quantity in each part of the solution. Does result this mean that the equation has no solution?

No. It obviously *does* have a solution. *Every* cubic has at least one real solution. Just look at the graph of this formula:

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Plot[x^3 - 15 x - 4, {x, -5, 5}, PlotLabel → "y = x^3 - 15 x - 4"]
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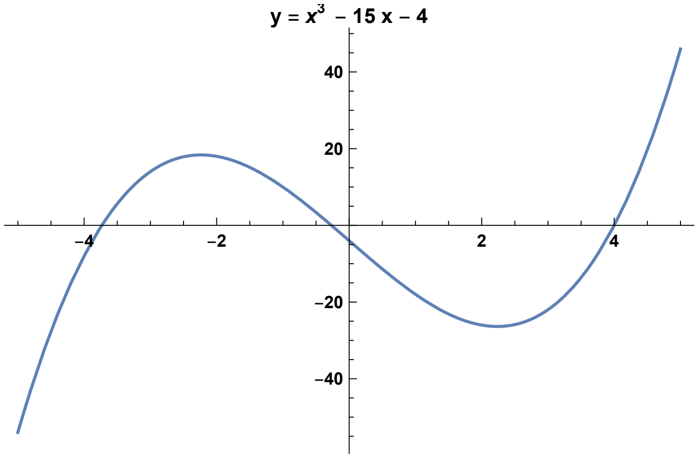

This formula should have *three* solutions. The solution farthest to the right seems to be +4. If there were any doubt, you can check and see that one of the solutions *is* +4.

$$4^3 - (15)4 - 4 = 64 - 60 - 4 = 0.$$

If the solution of the equation is +4, why did Cardano's formula give a result with two expressions involving the square roots of negative numbers? What went wrong with Cardano's Formula?

This is the question that confronted sixteenth century student of algebra, Rafael Bombelli.

(Rafael Bombelli, 1525 - 1572)

The short answer is that nothing went wrong. The slightly longer answer is that $\sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$ **is in fact** *equal to four.* That is far from obvious, which is why we should be grateful that Rafael Bombelli was a genius.

For it was Bombelli who stared at this formula, aware that one of the answers of the equation should be 4, and finally figured out how to make some sense of what he saw. It occurred to him -- we do not know how (although mathematician have reconstructed some plausible guesses) -- that the quantities $2+11\sqrt{-1}$ and $2-11\sqrt{-1}$ were in fact *cubes of other complex quantities*:

$$2+\sqrt{-1} \qquad \text{and } 2-\sqrt{-1}\,.$$

Try it!

$$\left(2+\sqrt{-1}\right)^2 = \left(2+\sqrt{-1}\right)\left(2+\sqrt{-1}\right) = 4+2\sqrt{-1}+2\sqrt{-1}-1 = 3+4\sqrt{-1}$$

$$\left(2+\sqrt{-1}\right)^3 = \left(2+\sqrt{-1}\right)^2\left(2+\sqrt{-1}\right) = \left(3+4\sqrt{-1}\right)\left(2+\sqrt{-1}\right) = 6+8\sqrt{-1}+3\sqrt{-1}-4 = 2+11\sqrt{-1}$$

And:

$$\left(2-\sqrt{-1}\right)^2 = \left(2-\sqrt{-1}\right)\left(2-\sqrt{-1}\right) = 4-2\sqrt{-1}-2\sqrt{-1}-1 = 3-4\sqrt{-1}$$

$$\left(2-\sqrt{-1}\right)^3 = \left(2-\sqrt{-1}\right)^2\left(2-\sqrt{-1}\right) = \left(3-4\sqrt{-1}\right)\left(2-\sqrt{-1}\right) = 6-8\sqrt{-1}-3\sqrt{-1}-4 = 2-11\sqrt{-1}$$

With this in hand, the strange result of Cardano's Formula reduces like this:

$$x = \sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}} = \sqrt[3]{\left(2+\sqrt{-1}\right)^3} + \sqrt[3]{\left(2-\sqrt{-1}\right)^3} = 2+\sqrt{-1}+2-\sqrt{-1} = 4$$

The square roots of negative one cancel out and the simple answer appears. This looks like magic, especially if you have spent hours staring hopelessly at the problem, increasingly convinced that it was created by the devil to torment incautious humans.

This strange result seems to suggest that the square root of a negative number **can form part of a meaningful formula**, provided the imaginary quantities in the expression cancel one another out before the final solution appears. We still don't know quite what $\sqrt{-1}$ means, but its appearance no longer implies instantly that a formula is meaningless. It can participate in the strange morrice-dance of algebra and lead ultimately to verifiable solutions.

Bombelli's result, wonderful as it is, is far from fully enlightening. Pleased as we might be that Bombelli could *guess* what complex number, when cubed, would give the particular quantity he was looking for, his success doesn't give us much help in solving *other* problems. Moreover, it doesn't give us much help in figuring out what the square root of negative one *is.* Do the imaginary numbers have any real meaning? Or are they bizarre brambles that need to be cleared away by *ad hoc* trickery? Was Bombelli's result a lucky accident? Or does it represent something valuable?

All *that* is the subject of the next lecture.

Appendix 1: How to Depress a Cubic Equation

Grant Franks

August 28, 2019

Begin with the general cubic:

$$x^3 + b x^2 + c x + d = 0$$

For *x* substitute $\left(y - \frac{b}{3}\right)$:

$$\left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d = 0$$

$$\left(y^3 - b y^2 + \frac{b^2}{3} y - \frac{b^3}{27}\right) + \left(b y^2 - \frac{2 b^2}{3} y + \frac{b^3}{9}\right) + c y - \frac{c b}{3} + d = 0$$

$$y^3 + (b - b)\, y^2 + \left(c - \frac{b^2}{3}\right) y + \left(\frac{2 b^3}{27} - \frac{c b}{3} + d\right) = 0$$

$$y^3 + \quad \ldots \quad + \left(c - \frac{b^2}{3}\right) y + \left(\frac{2 b^3}{27} - \frac{c b}{3} + d\right) = 0$$

Et voila, a depressed cubic.

For example, suppose you had $x^3 + 9 x^2 - 3 x + 2 = 0$. Substitute $x = \left(y - \frac{9}{3}\right)$:

$$(y - 3)^3 + 9 (y - 3)^2 - 3 (y - 3) + 2 = 0$$

$$(y^3 - 9 y^2 + 27 t - 27) + (9 y^2 - 54 y + 81) - (3 y + 9) + 2 = 0$$

$$y^3 - 24 y + 47 = 0$$

You can solve this (depressed) equation using Cardano's formula. Then from the values of y, find the values of x from the relation:

$$x = \left(y - \frac{b}{3}\right) \qquad \text{which means} \qquad x + \frac{b}{3} = y$$

Appendix 2: Lectures in the Series

Wednesday afternoon lectures, 3:15 pm in the Junior Common Room
Tuesday evening lectures, 7:30 pm in Room FAB 109

1. Say Hello to Your New Best Friend: $\sqrt{-1}$
(Wednesday, September 11)

2. The Bridge between Algebra and Trigonometry
(Tuesday, September 17)

3. Constructible Numbers
(Tuesday, September 24)

4. Building the Pentagon with Algebra
(Tuesday, October 1)

5. The Heptadecagon (17-gon)
(Tuesday, October 8)

6. Numbers and Meaning
(Wednesday, October 16)