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A GHOST

STORY

ROBERT SACKS

It has often been said in jest and in derision of a man considered to be a fool that he "has done nothing more than reinvent the wheel." I hope that during the course this talk I will be able to adequately articulate why it is that I would give my right eye only to be such a fool, but perhaps none of us is completely up to such a task.

Another way of saying the same thing is to sing that old tune:

If I could I surely would
Stand on the rock where Moses stood.

Now I do not wish to deceive myself into thinking that if only I could stand on the right rock I too would have seen what Moses, or Euclid, or Apollonius saw. I would not have. I have not such sharp eyes. On the other hand, it seems to me that to understand anything means to understand how it might have become visible to a human eye. Perhaps more than any other scientist Einstein seemed to have had a full grasp of the problem. Constantly he was writing papers and articles which deal in whole or in part with the question of how he came to the notion of Relativity. For him answer are wonderful things. They are the things we are all looking for, but he also saw that they have one potentially fatal drawback. Answers tend to erase the question. In doing so they erase themselves as answers and harden themselves into prejudices and facts to be taken for granted. On the other hand, it must be pointed out that he published many quite different accounts and that many were even contradictory. For him it was important to see the several different ways a thought might have come before a human eye. The question of which of these ways first came to any particular human being is a question of mere history and of no interest to him or to us in this evenings lecture.

PART ONE

Plato once recorded a great conversation which is said to have taken place between Socrates and the slave of a by then defunct general named Meno. The passage I have in mind begins when Socrates says to Meno:

Suppose that you call one of your numerous attendants, that I may demonstrate on him.

Meno answered:

Certainly. Come hither, boy.

Thus, we are to imagine a large number of people milling around, some listening, others perhaps not listening; each smiling his own smile or scoffing his own scoff. Plato does not tell us who they were, or describe their several reactions in any of the received manuscripts, but there is a vast labyrinthine tradition from which I believe I have been able to reconstruct, or to remember some fragments of that day. Some say that Euclid was there, but I tend to have my doubts. The timing is all wrong. Euclid lived and thought many generations later. For him the notion of pre-stated postulates was already out there, like a mountain or a blade of wild grass, or like an apple hanging on a tree. How easily that last example came off my tongue while I forgot that there are no apples worth the eating given by nature, but that each of them comes from a grafted tree. Each apple, then, is a product of that original insight into the possibility of grafting, an insight in which I have no share, although I mistook myself and thought I had all. What I had taken for granted as just being there had only been there because of another's labor. And so it may have been for Euclid; and so it surely is for his modern reader. And so perhaps Euclid was not there. At any rate someone was there and for all I know it was Euclid's grandfather. For the moment, however, let us continue to call our nameless friend by the name Euclid.

Socrates then asked Meno:

He is Greek, and speaks Greek, does he not?

To which Meno replied:

Yes, indeed; he was born in the house.

Euclid was pleased. "If they both speak Greek," he said to himself "they will not only be able to speak about geometry, they will be able to speak geometry itself. The boy, of course, will have to learn many strange new words, words not found on the every-day tongue, but they will be Greek, and so he's off to a good beginning."

"No," thought Descartes, "No, the fact that he can speak Greek just doesn't sound all that important to me. Ever since that fateful afternoon when I first saw that I had to take more seriously and make the center of my concern that curious relationship which I had originally no more than noticed between algebra and Pappus's concept of a locus, human thought and its relation to what it perceives has taken on a wholly new cast. If the human mind sees itself as radically other than the world it perceives or imagines about itself, and forces itself to think entirely within its own terms, it is capable of a kind clarity and distinctness of which no man has ever dreamed. A circle with the fullness of its roundness implies divinity to some and simple boredom to others but the clarity of the equation is the same to all. Mathematics cannot be spoken in Greek or Latin or in French, or in German. Mathematics can only be written in mathematics itself." And with that off he went to sit by his stove and work things out, paying no attention to the fact that the whole of his new enterprise rested on a decision he had made in normal every-day French.

Leibniz gave a complaisant kind of smile and thought: "Descartes should never have walked out. He's right, of course, Mathematics must be written in Mathematics; but it must also be spoken in Greek and in Latin and in French, and in German, yes and in Chinese too. Those who demand clarity understand that each language and in fact each person, precisely because they are who or what they are, will have a slightly different view even of what addition is. They say: 'one plus one **is** two' or 'one and one **are** two', or even 'one and one **make** two' and sometimes even 'one and one **makes** two'. Each one of these expressions a slightly different understanding of what is involved in the act of addition. That is why they prefer the equation. However, Descartes's clarity comes at the price of narrowness. It fails to see that each of those views reveals only one facet of the whole; a whole which no man may ever see with Cartesian clarity."

Socrates and the boy continued their conversation:

Soc: *Tell me, boy, do you know that a figure like this is a square?*

Boy: *I do.*

Soc: *And you know that a square figure has these four lines equal?*

Boy: *Certainly.*

Soc: *And these lines which I have drawn through the middle of the square are also equal?*

Boy: *Yes.*

Euclid burst in. "But Socrates, the boy doesn't really know any of those things. Why the construction of the square alone requires five postulates, five common notions, seventeen propositions,[1] and God knows how many definitions!"

"No, my dear friend," said Socrates," the boy knows many things. He knows, for instance, that if two sides of a triangle are equal the angles at the bottom are equal as well. He also knows that that truth has nothing to do with those silly little lines you've drawn underneath, but comes from something he fully grasps and to which those more sophisticated than he, give the name 'symmetry'. These things he knows as well as you know your five postulates. How well that is we shall not at present ask."

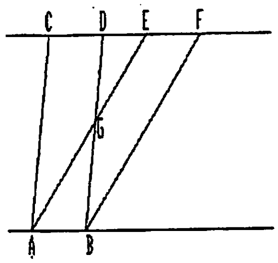

Figure 1

"He also would know that if there are two sets of lines lay along each other as in Figure 1, then AB equals CD, and AC equals BD, but he might not know at first that ABCD equals ABFE. However, if the lines AC, CE, and EA should happen to fall together in his eye as a triangle, then he might see that the triangle ACE is equal to the triangle BDF because it's the same thing just slid over. And if he were curious enough and had the kind of eye that could hold both the triangles and the quadrilaterals in mind at the same time, he would be in for quite a thrill.

"And so you see, my dear man, there is a whole world of, let us say, five hundred things that the boy already knows and from which he will go on to discover much more. I pick the number five hundred partly at random and partly as a private joke."

Euclid was not much impressed by Socrates' speech and thought that no science could be founded on such beginnings. But for the moment at least, he held his peace.

Then Socrates and the boy went on to rediscover the double square. But did the boy need Socrates? Socrates was not always there. There must have been someone who found it on his own.

The question the arises: "Can I find a spot from which the double square might have become visible to a human eye, or must I too forever rely upon Socrates?

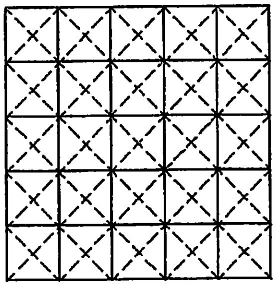

Figure 2

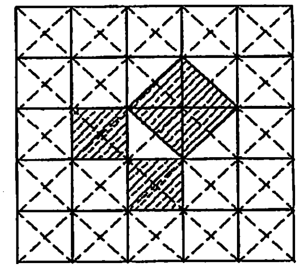

Figure 3

In one sense the solution to our problem is not so far. Who has not stared at a tile floor, as in figure 2, watching the patterns dance as the lines form themselves first into one group and then into another? Some may even have seen it as we have marked it in figure 3.

Surely many eyes saw that square, but it must have taken an extraordinary man to see that it was something worth remembering and of being concerned about: even worthy of telling ones neighbor. Perhaps that was because in some way he realized that when he said they were equal he meant something quite different from what Homer meant when he wrote the line:

> *When they had done all this and had prepared the feast, they ate it, and every man had his full and equal share,*

This equality felt deferent because it did not depend upon the skill of the stone mason but merely upon the pattern or eidos that he had intended. For that reason it would always be there, even for the neighbor, even for all generations to come.

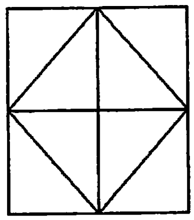

Figure 4

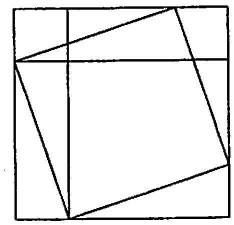

Figure 5

There is another thing that calls forth wonder. There are those rare individuals who whenever they see something ask themselves: "Is this a simple truth, or is it a minor case of something larger, stranger and even more wonderful than itself?" Some such a one must have asked himself "What if the square had no longer been a square, but a rectangle instead?" He redrew the figure as best he could, and discovered the square on the diagonal to be equal not to twice the rectangle, but rather to the sum of the squares on its two sides, since each in its own way was equal to the large square minus four times the triangle.

Now we must go back and ask ourselves why Euclid was so distraught. What did Euclid, or whoever it was, see that caused him to abandon that world which Socrates and the slave boy held in common so comfortably, and to replace it with five postulate all laid out and agreed upon beforehand? Of what was he so afraid? Perhaps it was something like this:

Almost since the beginning of time carpenters and hence geometritions have known how to bisect an angle. Then, according to Pappus, one of the Ancients proposed the following construction for trisecting an angle.

Given angle ABC, it is required to trisect it.
Let AC be perpendicular to BC and complete the parallelogram ACBF, and extend AF.

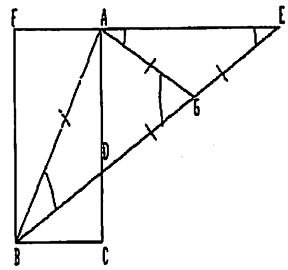

Figure 6

Let BF be drawn in such a way that it intersects AC at D and AF extended at E making DE equal to twice AB, and let G be its midpoint.

Now since DAE is a right angle, G is the center on the circle passing through points E, A, and D.

Thus, DG=GE=GA=AB.

Therefore
ABG = AGB = GAE + AEG = 2AEG
or ABG = 2 AEB
but DBC = AEB
Therefore
ABC = 3DBC
or
DBC = 1/3 ABC

Could you all follow the proof? Do you then all agree that I have succeed in trisecting the angle ABC? No? Well, I guess we'll have to take a vote. Will those who accept the proof indicate by saying aye? Opposed? The problem, of course is whether you inclined to allow me to draw the line ADE. Many were so inclined, and to be truthful, I do not know where I would have stood had Euclid not been our tutor.

But what does it mean to have voted on the validity of a mathematical proposition? Have we not just destroyed even the possibility of any meaningful mathematics? Now I am prepared to read Euclid's reconstruction of geometry, artful and hence as unnatural as it may be, unless I am prepared to see the crumbling away of mathematics in it's entirety. This will require replacing our common world of the five hundred which has always been out there and in which we have always lived, with a world based on five pre-stated postulates laid down by one man and agreed to by all. It may also mean starting with that kind of definition which Meno was looking for and in which Socrates saw so much that was doubtful. There may be no other way - and the beauty of what has been built on this edifice is surely undeniable.

Euclid himself, it must be pointed out, was not wholly committed to these new beginnings. For instance, he still assumes without postulate that if a straight line contains at least one point laying within a circle and at least one point lying outside the circle, then it contains at least point lying on the circle.

Do I think that it was precisely the trisection of the angle that moved our Euclid, whoever he was, to found geometry upon a limited number of pre-stated axioms and postulates? Well possibly not. It could equally have ben the discovery of irrationality. But, so far as I can see, only some experience such as we have just gone through could have given rise to the issue. As far as the exact proof is concerned, not only is it unclear that such merely historical knowledge is available, its value in most cases is even dubious. Far more important is our realization that our mode of thought itself is the hard earned fruit of human thought and not a thing to be taken for granted as given.

The Meno began when Meno asked the question:

> *Can you tell me, Socrates, whether virtue is acquired by teaching or by practice; or if neither by teaching nor practice then whether it comes to man by nature, or in what other way?*

Most of our thoughts such as the need for pre-stated postulates, are all of these. They are like ghosts, ghosts of the thought and concerns of other men that flit from mind to mind, taking up residence in houses in which they were not born, and masquerading as living beings. Home-grown thoughts come to be within that particular but fuzzy horizon we call our mind. They trail off into the periphery of our horizon, and if pushed, can find those chinks in the wall which lead beyond any horizon, back to those foggy primal inklings that first gave rise to our full blown thought. The ghosts of other mens thoughts reach out for there own craggy horizon, and not finding it, they can only return to the center they where congeal into a hard rock of sedimentation. Many magnificent and even beautiful structures can be built on that rock. Witness modern science. They are no less the product of genuine insight than there predecessors, but they rest upon the ghost that has become a rock, a living thought in the teacher, but in the student, solid and unquestionable.

This is the world which Descartes wished to set aside by a simple act of universal doubt. But the ghosts bury themselves too deeply into our souls to be whisked away in the twinkling of an eye. Ghosts can only be exorcised by bringing them back to life as living thoughts being thought through as objects of serious concern.

At the end of their conversation when most of us are firmly convinced than the boy has finally understood how to find the double square, Socrates surprises us by saying:

> *At present these notions have just been stirred up in him, as in a dream; but if he were frequently asked the same questions, in different forms, he would know as well as any one at last?*

Euclid could no longer contain himself and finally spoke up. "Now, Socrates, you see where all of this has lead. When I prove a proposition it by God stays proved."

The two spoke no more on that day, but as they parted Socrates was thinking about what Leibnitz had said, while Euclid was concerned about the man who thought he had trisected an angle.

PART TWO

The ancients regarded his work on the Conics as one of Apollonius's minor works, and I must say that I tend to agree with them although I consider the Conics to be one of the two or three most insightful book on mathematics I have ever read. They thought of him primarily as an astronomer, and that his Great Work was his now lost work in astronomy. To understand this we must take a brief look at the subject as it was before he came along.

As you know, the Babylonian priests had diligently kept daily records of the motion of the planets over a period of many centuries. To me it is incomprehensible that men should have devoted themselves to such a task when no discernible pattern could have emerged for generations? What started them? What kept them going? Perhaps it was the conviction that wherever a hint of order shines forth more will appear if one is dogged enough in ones looking.

In the case of the wanderer Venus, they found that while the star more or less kept pace with the sun, it was sometimes to the north, sometimes to the south. First it would run ahead of the sun, that is to the east, Then it would slow down till it began to lag behind the sun. Then it would speed up and start the whole thing over again.

The only problem was the distances and times of these sallies kept changing with no apparent order to them. Now you have to remember these priests were men who had no tradition of science behind them, and yet the continued looking and recording till they were satisfied that this highly erratic series began to repeat itself.

While the Babylonians, so far as I can see, did have some notion of the stars as eyes set in wheels that turned because the spirit was in the wheels, they had nothing that would address the back and forth motion of Venus. This was left for a Greek named Eudoxos.

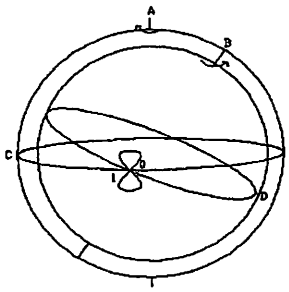

Figure 7

He set the axis A into a sphere that moved with the sun, so that if nothing more were done, the star, at point 0 would move with the sun. However, he added two more motions. One moved clockwise around the axis A, while the other moved at the same speed around the axis B, in the contrary direction. As you can see, if the two poles were set along the same straight line, the two motions would cancel each other out and the star would remain at the point 0. If, however, the poles are inclined to one another, the motion towards D will pull the star somewhat to the south, but now the motion to the east will no longer overcome the motion to the west, and the star will end up at point 1. It will then go on to make a quasi figure eight. As you can see, the star will sometimes be above the sun, sometimes below. At times it was ahead of the sun, at times behind. However, according to this construction, the star would have to complete exactly two back-and-forth motions for each up-and-down motion, which is very far from the truth. Eudoxos only succeeded in accounting for the kind of thing that happens, and made no pretence to accuracy whatsoever.

Another way of saying these things is that for Eudoxus the phenomena, the things to be accounted for, were the phenomenal things, things such as 'retrogradation" "eclipses", and the like whereas for Ptolomy the phenomena were the ordinary things. That meant accounting for the appearance of each day according to its day. If that work had been done well, the phenomenal things would take care of themselves. How this change might have taken place, we have yet to see.

With that in mind, we must next turn to the Conics of Apollonius.

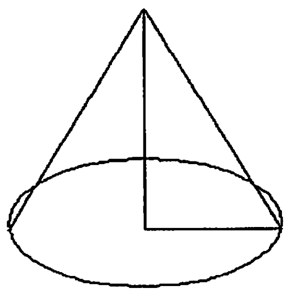

Figure 8

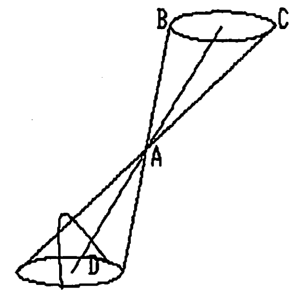

Figure 9

Prior to him the cone was generated by revolving a right triangle about one of its legs. He, on the other hand, defined it as the surface generated when a line such as AC is rotated in such a was that the point A remains constant while the point C moves along the circle CB.

I believe, by the way, that what originally sparked his interest was his need, as they say, to build a better sundial. The circle BC was the daily path of the sun, A was the tip of the pointer, the cutting plane was the plate of the sundial, and the conic section was the path of the shadow to be calculated for some given day. But all this was soon forgotten, and we can see the act of thought at work when we see how fascination replaces need.

In Book One Prop. 36 Apollonius proves that if from a point on a conic section a tangent and an ordinate be both drawn to the axis, then, in terms of our figure:

AB:AB'::A'B:A'B'

or the point A cuts the line BB' on the outside in the same proportion that A' cuts it in the inside.

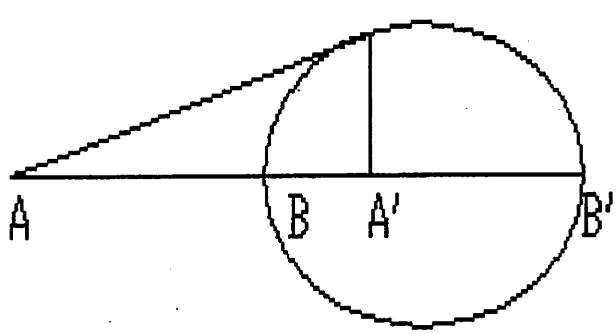

Figure 10

For many reasons this ratio is called *the harmonic ratio*. For us the harmony lies in the fact that if we consider the line BB' and take the point A' on it, if A' is say seven times closer to B that it is to B' A is also seven times closer to B that it is to B'. In that sense, the two points are *in harmony*. It should be clear that as A' is taken closer to the center, AB can become as long as you please.

The first question that always enters my mind when I read something like Proposition 36 is, "How did he smell out that such a thing even might be true? Where was he standing when such a possibility occurred to him? But of course that is merely a folksy way of putting the question and as stated of no more than historical interest. The real question is "How might such a possibility have become visible to the human mind as such?" Of course, "to become visible" means "to become visible" from within a specific, and hence a historically specific horizon." In what sense this is true I hope to make somewhat clearer in the course of this talk.

In the special case of the circle, it is pretty clear that

since AOC ≈ A'OC

AO:CO::CO:A'O

or AO:B0::BO:A'O

or $AO,A'O=BO^2$

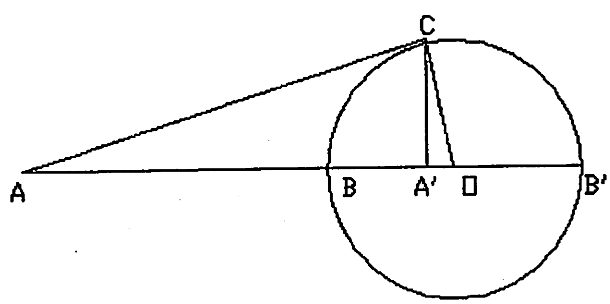

Figure 11

which is what he proves for all the Conics in the following proposition I:37. The harmonic ratio, I:36 for circles then easily follows from our version of I:37. For:

AO:OB::OB:OA'	AO:OB'::OB':OA'
AB:OB::A'B:OA'	AB':OB'::A'B':OA'
AB:A'B::OB:OA'	AB':A'B'::OB:OA'

AB:A'B::AB':A'B'

All this about the harmonic ratio, of course, is still based on the hint we took from Apollonius. We still do not know what attracted him to this proposition. That is to say, what made him think that it might hold for all Conics in spite of the fact that AOC will no longer be similar to A'OC? The fact that the point C was no longer plays any role in the final statement lead me to construct a number of ellipses on the same major axis like this:

This lead me to what I've grown to call the nutcracker view of the ellipse. That, I thought, was a lot of fun and might be of use one day but it just didn't quite smell like a beginning: something a keen eye could have picked up just because it was there. As I continued to gaze at the figure, however, it suddenly transformed itself into a sphere resting in a cone. This was enough to guarantee that the tangents to the ellipses would all meet the diameter at the same point. This, it seemed to me, was something a eye keener than my own might have picked up and I saw that a mind thirstier than mine might have been sufficiently struck by what it had seen to search out that ratio by using the case of the circle. All this seemed even more likely to me when I remembered that Apollonius was an astronomer, and as such would have spend late hours drawing many such figures, hence knowing that spheres are drawn by drawing ellipses.

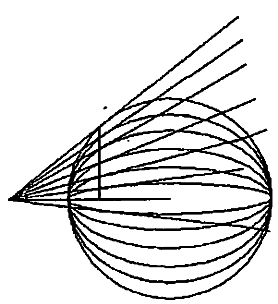

Figure 12

A mind like that of Apollonius is the true meeting of Dionysus and Apollo. Look at any one of his propositions. Imagine the wildness of his thoughts as he searched for what might be true. Then consider the sobriety with which he labored to reestablish these truths on good Euclidean grounds..

Let's look back for a moment to see where we are. We have a Euclidean proof that in a circle AO,OA' = OB^2, from which we derived the harmonic ratio. We also know with Pre-Euclidean certainty that these things are also true of the Ellipse, but we have no Euclidean proof. We are also sure that if we ever find such a proof, the hyperbola will either come along quietly, or the something equally interesting will happen.

We must now begin to look for such a proof, but before doing so, let us compare the normal Euclidean proof with the normal Apollonian proof. Most of Euclid's longer proofs refer back to many earlier proofs, whereas Apollonius rarely refers back to more than one earlier proof except for those that deal with the handling and transformation of ratios. To that extent, the art of Apollonius is different from the art of Euclid. Hence, his art of searching may also differ from that of Euclid.

The first thing he must have noticed is that the proposition AO,OA' = OB^2, so easily proved for the circle cannot be quite so easily proved in the case of the ellipse. As it turned out, he was thus forced to turn the whole thing round by establishing the harmonic ratio on its own and then derive 37 from it. This longer route, he found, required the whole of Book One up to that point. In fact, it might be wise to analyze Book One in the light of its having been the product of his search for that proof. I sometimes call it "rumbling through Apollonius's wastepaper basket." And so at this point I shall leave you to follow his backwards journey to see how he managed to arrive at his beginning. Before going on, however, let us take one more look at the figure for I:36 for the case of the circle as Apollonius himself may have done.

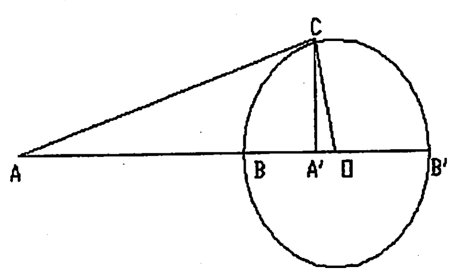

Figure 11

Memory is strange and awesome. Augustine speaks of ... *the plains, and caves, and caverns of my memory, innumerable and innumerably full of innumerable kinds of things*. He thinks of it as a place where .... *images are with an admirable swiftness caught up, and stored as it were in wondrous cabinets*. But the conceit of memory as a place or mansion where things are kept on shelves or in bins or on disks like tokens or little slips of paper waiting to be picked up by a passing hand never seemed right to me. Memory has always seemed to me more like an activity than like a thing: more akin to a low level thought than to the pressed rose itself. At some level we must always be thinking every thought we have ever thought, seeing everything we have ever seen, hearing everything ever heard. Memories often seem more to come to us than we to them. We can almost sense this activity going on when we say "The word is on the tip of my tung."

Apollonius, as we remember, first became attracted to the study of Conic Sections, through his interest as an astronomer in building a more accurate, more intelligible sun dial, or so it seems to me. But from what Ptolemy says of him I think something quite different began to happen as he took a second look at the diagram for Proposition 38. Those old memories of Eudoxus began to stir more rapidly. But had they not been alive in him at some low level they could not have come looking for him nor could he have gone looking for them. If point A were the earth and if the point O were to move evenly with the mean position of the sun, then the planet Venus would be free to travel at it's own speed around the circle BB'. This way one could tune the motion to achieve a kind of accuracy Eudoxus never dreamed of. From what Ptolemy says as well as what we can see in *The Conics* of his general interest relation of inside to outside, it is clear that he knew that the earth could be understood to be at the point A' as well, especially in the case of the outer planets. And if he knew that much, God knows what else he knew.

Normally we think of men as having questions for which they then go on to look for answers. I should like to suggest that almost the opposite it true. Questions only arise when we almost have an answer. Neither order nor chaos can give rise to a question. As it was for Job so it was for Apollonius and so it is for all men. There are things that make too much sense to make no sense, and yet they make no sense. These are the things that give rise to questions. In the evening and in the morning is wonder born.

PART THREE

In Proposition Sixteen of his **Theory of Parallel** Lobachevski writes: "In the uncertainty whether the perpendicular AE is the only line which does not meet DC we will assume it may be possible that there still other lines, for example AG that do not meet DC how far soever they may be extended."

We read, but we do not understand - unless we, like the author, have already felt that uncertainty for ourselves. We could read on, and if we did, we would find ourselves in the middle of a strange new world. Soon, perhaps all too soon, we would begin to be able to navigate the seas of this new world and even discover some of its as yet uncharter-ed islands. But there would be a gap in our memory. It would be as if we had arrived in the new world by magic carpet and never travelled the road from home by foot. We would never have seen the breech or chink in the horizon of our old world which caused our fathers to go beyond them.

I remember that when I was a Junior someone, it may have been Jasha Klein, related to me the a line which has be attributed at one time or another to almost everyone who lived in the Seventeenth Century. It says: '*The ancient were giants and we are but midgets, but since the midgets stand on the shoulders of the giants, they can see farther than did the giants.*" I felt that it was very true, but that those who were most fond of saying did not understand it fullest conse-quences. There was one thing which the giant saw which the midget could not see. He could not see the ground upon which he himself rested because the giants head was in the way.

Our path begins with a remark made by a man named Proclus, but to understand it we must remind ourselves of some things in Euclid.

Proposition 27 of Book One reads:

If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

Proposition 29 of Book One reads in part:

A straight line falling on parallel straight lines makes the alternate angles equal to one another.

As you no doubt remember, Proposition 29 relies on The Fifth Postulate whereas Proposition 27 does not. I remind you, but only parenthetically for the moment, that it does however rely on Proposition 16.

Proclus was outraged. In the commentary to Proposition 29 he says: *...not everybody admits that this generally accepted proposition is indemonstrable. For how could it be so when its converse is recorded among the theorems as something demonstrably?* He had already said of The Fifth Postulate: "*This ought to be struck from the postulates altogether. For it is a theorem.... and the converse of it was proved by Euclid himself as a theorem.*"

For Proclus there was a fundamental asymmetry in the foundations of geometer which, if not corrected would soon cause its downfall. To avoid this downfall it was absolutely necessary to come up with a proof of the Fifth Postulate. To understand ourselves and where we stand, we must follow that pursuit, but first we must add a long footnote.

FOOTNOTE

Proclus was right, but only to a certain extent. There was a certain asymmetry in Euclid, and it did lead to his downfall from the unique position he once held. However the asymmetry was not the one Proclus had in mind.

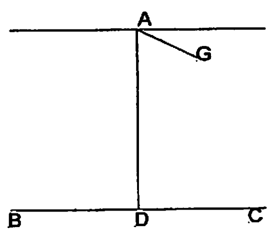

FIGURE 13

As we pointed out before, Proposition 27 relies on Proposition 16 which presupposes that two straight lines cannot have more than one points in common. It is easy to see that this is in fact the counterpart of The Fifth Postulate. If we look at Figure 13 and ask ourselves "How many straight lined can be drawn through the point A which do not cut BC?" We might begin by answering "either one, or many, or none." The notion that there could be many is negated by the explicitly stated Fifth Postulate, but let us examine the notion that there might be none. Conceder, then, the perpendicular AE. Will it meet BC on the right or on the left? Symmetry would say neither by itself will do, and so if it must meet then it must meet in both. The true asymmetry lies in the fact that there are two symmetrical postulates, one explicitly stated, i.e. The Fifth Postulate, the other, that two lines can meet in at most one point, left unstated. But I repeat, Proclus seems not to have noticed that. Why did Euclid do that? From something Proclus says I suspect that the notion that there were lines that seemed to converge but did not meet was new and paradoxical and that Euclid found it necessary to postulate that no such fate was in store for straight lines. On the other hand ever Homer knew that it took circle and a straight line to meet at more than one place.

END OF FOOTNOTE

A thousand years went by, during which time countless numbers of people tried their hand at proving The Fifth Postulate, mainly by *reductio ad absurdum.* Many strange and absurd thing were proven, but each time is was shown that the theorem only contradicted a theorem in Euclid which had itself been established by means of The Fifth Postulate. Strictly speaking none of these admittedly bizarre propositions could not be disproved on the basis of the first four postulates alone. Nothing had been gained. To put it metaphorically, it was as if many chimerical house had been built with the intention that they fall, but none of the fell. And yet no one ever took any of them to be more then fantastical.

But in 1733 Girolamo Saccheri published a book *EUCLID AB OMNI NÆVO VINDICATUS* or **Euclid Freed of Every Fleck.** I noted the curious name, but it rang no bell for me at the time.

Unlike those who came before him, Saccheri was fully aware of the fact that Proposition 27 rested on the hidden postulate. He was, therefore, no longer troubled by the supposed problem of asymmetry.

He, in effect, begins by replacing the Fifth Postulate with two hypotheses. The first, which he calls *Obtuse angle Hypothesis* and which states:

> **Through a point not on a given line there cannot be drawn any straight lines which do not cut the given line.**

The second, which he calls *The Acute Angle Hypothesis* states:

> **Through a point not on a given line there can be drawn more than one straight line which does not cut the given line.**

Ostensibly, he is out to prove that both the **Obtuse angle Hypothesis** and the **Acute Angle Hypothesis,** as he says are "Absolutely False."

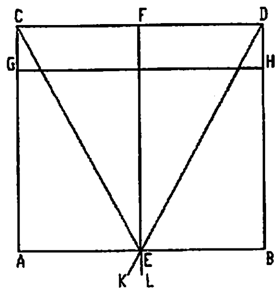

Figure 14

In his fundamental figure. AC & BD are taken equal to each other and at right angles to AB. E & F are the midpoints of AB ad CD. First he proves:

ACD = BDC

And that. The angles at E & F are right.

Then he proves:

IF	THEN
ACD <=> a right angle	CD >=< AB
CD >=< AB	ACD <=> a right angle
ACD <=> a right angle	AGH <=> a right angle
If a hypothesis is true in one case	It is true in all cases.
ACD <=> a right angle	KEL >=< EDB

Then in Proposition Fourteen he proves:

The hypothesis of the obtuse angle is absolutely false, because it destroys itself.

However, he explicitly uses Euclid 1:17 which he knows to contain the hidden postulate and which therefore makes it useless for proving the postulate. The very next proposition, Proposition 15 reads:

By any triangle ABC, of which the three angles are =, >, or < 2 right angles is established the hypothesis of the right, obtuse, or acute angle

As we suspected, *The hypothesis of the obtuse angle* is as well and alive as ever, and the same fait awaits *The hypothesis of the acute angle* later in the book.

I now believe I have understood that rather strange title. Does one vindicate Euclid of every fleck by proving The Fifth Postulate thus showing him to be a fool for having postulated it, or by showing the he was right in forging such a postulate because it could not in fact be proved?

Unlike those before him, Saccheri has a global understanding of each hypothesis. The fantastical houses had pulled themselves into two distinct though fantastical cities. The unity of each city lay in the fact that he could guess beforehand which building would stand in which city. It was this unity which made the fantastical less fantastic, even thinkable. However he had no proof of consistency, but only a pre-Euclidean sense of their unity.

Lobachevsky, by establishing a trigonometry, proved that the new geometry is consistent if our number system is consistent. However, the consistency of our number system still rests on our pre-Euclidean sense of unity, and Goedel's Theorem seems to show that it always will.

And so it is that no one ever went out in quest of a promised land unless he had already had an inadvertent glimpse from off high on a mountain top.

End note

1. 01, 02, 03, 04, 05, 07, 08, 11, 13, 15, 16, 23, 26, 27, 29, 31, 34, 46,