

ON INTERLOCKING SIMILAR OR CORRESPONDING FIGURES AND ORNAMENTAL PATTERNS OF CUBIC EQUATIONS

In recent years, studies concerning the application of geometry in Islamic art and architecture have gained particular impetus, as a fairly large number of original documents on the subject, both written and drawn, have surfaced and become available to researchers. The written material is mainly the work of mathematicians. The drawn material consists of architectural and decorative designs on stucco plates and fragments and sheets or scrolls of paper, unaccompanied by any explanatory text. One source, however, has both: entitled *Fi tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (On interlocking similar and corresponding figures; referred to hereafter as *Interlocking Figures*), it is a compilation of drawings of ornamental geometry accompanied by anonymous explanatory texts in Persian. It is of particular significance because it provides a link between written and drawn material on the subject.¹ The only copy so far known is preserved in MS Persan 169 in the Bibliothèque Nationale, Paris, a compilation of twenty-five works on mathematical subjects, mainly practical geometry.² The fact that *Interlocking Figures* (fols. 180r–199v) follows one of the Persian translations of a treatise by Abu'l Wafā³ al-Buzjani (940–998), *Fīmā yahtaju ilayhi al-sāni^c min a^cmal al-handasa* (On what the artisan requires of geometrical constructions; hereafter referred to as *Geometrical Constructions*) in the same manuscript (fols. 141v–179v) has led several scholars to assume that it was meant to be an appendix to Abu'l Wafā³'s work.³

Historians of art and architecture agree that *Interlocking Figures* is an important source for the insight it provides into how artisans used applied geometry. Its mathematical content, however, has had mixed reactions. Bulatov judged it to be a kind of textbook explaining the basic principles of geometric decoration and reflecting the state of applied geometry in its time.⁴ Chorbachi extolled it as representing a much higher and later stage of development than Abu'l-Wafā³'s *Geometrical Constructions*.⁵ Necipoğlu assumed that its author was a mathematician who had practical rather than theoretical training in geometry.⁶ Relying on Bulatov's translation and pointing to its deficiencies as a work on geometry, I first

identified the author as an artisan who had a superficial acquaintance with geometry, rather than a mathematician who had been occupied with the ornamental arts. Recently I was finally able to study the original manuscript, however, and I now feel obliged to revise that assessment and rectify points raised earlier.⁷

THE GENERAL LAYOUT OF *INTERLOCKING FIGURES*

The document is a collection of geometric constructions loosely grouped together according to topic rather than a treatise on geometry which requires a predetermined structure, an argument, or at least a logical sequence. In *Geometric Constructions*, for instance, Abu'l Wafā^c, following the title page on which he dedicates his book to the Buyid ruler Baha^c al-Dawla, lists the subjects of the thirteen chapters of his book, starts with the instruments and basic procedures of geometry, and then proceeds to the constructions of regular figures, superimposed figures, divisions of triangles and squares, dissections and compositions of squares, unequally sided figures, tangents, and finally polyhedrons.⁸ In *Interlocking Figures*, on the other hand, some constructions share a common topic and follow in sequence, but some appear in isolation; the topics seem to have been selected at random without any underlying structure or reason to link them in order. Some constructions pertaining to a particular topic can be found — with no explanatory text — next to a construction having nothing at all to do with the topic. Even the title appears as a vertically placed annotation on the first folio without any preliminary, giving the impression of its having been added as an afterthought. The layout of the folios themselves reflects the same absence of organization: explanatory texts — and in some cases constructions — are not always horizontally laid out: some are inclined in various directions, placed vertically, or follow a curve, or are even upside down. In short, the generally haphazard display in *Interlocking Figures* makes it difficult to identify it as a treatise on geometry conceived of, and deliberately put together by, a sin-

gle author. It rather gives the impression of a collection of notes, in some cases hastily taken. If that is the case, was the compiler a mathematician involved in ornamental arts, or an artisan familiar with geometry, or someone else?

While describing an instrument used, it is said, for certain constructions involving conic sections, the writer refers to himself as a scribe (*kātib*) and admits his ignorance of such procedures. Two alternative explanations are that the *katib* was the copyist of the manuscript who added a comment of his own — which would not have been unusual in medieval times — or he was the compiler who had taken the notes. The former offers us the convenience of ascribing all the mistakes to the copyist, but does it explain all the inherent deficiencies? Before passing any judgment on this point and especially on the source of the compiled material, its mathematical content needs to be analyzed.

The practice of having a scribe take notes during a discussion on geometry and then compile them as a work on geometry was not unknown. Ca'fer Efendi, the author of *Risāle-i mi'māriyye* (1614–15), who had been in the service of the Chief Ottoman Imperial Architect Mehmed Ağa for more than twenty years, tells us:

Because we have been connected with him [Mehmed Ağa], for the most part closely, for many years until the present time, when certain subjects concerning the science of geometry were being discussed this humble servant took and wrote down everything. In accordance with this, [I] set down and composed a treatise (*risāle*) concerning the science of geometry Regarding that which is called the science of geometry, in this age, if the science of geometry is discussed among architects and learned men⁹

It appears that Ca'fer's treatise, which cannot be traced, was a compilation of notes taken during a series of special meetings between the artisan-architects and mathematicians over a span of about twenty years. Since he informs us that he had studied the sciences (*ʿulūm*) in Istanbul and had become a client of Mehmed Ağa, before the latter's appointment as the supervisor of water works (*su naziri*), he appears to be a professional scribe. According to the literary sources, scribes were expected to be educated in the general sciences, literary and calligraphic skills, practical applications of arithmetic, and mensuration — a branch of practical geometry¹⁰ — but constructive geometry was not part of their training.¹¹

Meetings of this sort, which I will call, for want of a better word,¹² *conversazioni*, that is, gatherings of artisans and mathematicians to discuss problems concerning the

application of geometry to architecture and the related arts, seem to have been quite common affairs in the Islamic world. Abu'l-Wafa³ tells us that he took part in several of these *conversazioni* between artisans and mathematicians in Baghdad. We also deduce from an untitled treatise of Omar Khayyam (ca. 1044–1123) that he attended a *conversazione* in Isfahan, which inspired him to write his treatise. The gatherings at the construction sites in Samarqand to which Ghiyath al-Din Jamshid al-Kashi (d. 1429) and other mathematicians contributed can also be considered *conversazioni*. All these references suggest that in major centers, where sciences and arts flourished, *conversazioni* acted as the medium through which mathematicians and artisans exchanged ideas.¹³

The references in *Interlocking Figures* to "some of the artisans" (*ba'ḍi az ṣunnāʿ*) and to "some of the experts in geometry" (*ba'ḍi az aṣḥāb-i handasa*) not only points to the collaboration between artisans and mathematicians, but also give the impression that the person who wrote it belonged to neither profession. One of them suggests a typical *conversazione*: "Master craftsmen (*ustādān*) have inquired about the pattern (*ʿaqd*) in question, Abu Bakr al-Khalil investigated it in several ways and explained various solutions, one of which is constructed."¹⁴ Abu Bakr al-Khalil's name was later cited again, with the additional appellation, *al-tajir* (the merchant). The Paris manuscript also contains two undated works on practical geometry and applied mensuration by this otherwise unknown mathematician whose name there has been given the additional appellation, *al-rasadi* (the astronomer). One thus suspects that the *Interlocking Figures*, like Ca'fer's treatise, may be the record of a series of *conversazioni* between artisans and mathematicians.¹⁵

Abu'l Wafa³'s account of a *conversazione* that he attended in Baghdad provides insight into the actual collaboration of mathematicians with artisans. According to him, "dissection and composition" (*qismat wa taʿlif*) was a method widely used by the artisans, but they usually committed gross mistakes in its application. In this particular *conversazione*, a question was raised about constructing a square from three squares, a problem that amounts to finding the square root of three. When a mathematician proposed an algebraic solution, none of the artisans were satisfied; they wanted to dissect the three squares into parts and then compose them in a single square. The artisans themselves offered two solutions. Saying that these constructions were conceived of by somebody who did not have any knowledge in crafts and geometry, Abu'l-Wafa³ patiently proved that their methods of dissection-and-composition were wrong, and

then proposed an elegant solution to composing a square from three dissected squares. Afterwards, he offered a novel proof of the famous Pythagorean theorem by dissecting a square into four congruent right triangles rotating around a central square and then composing these parts in two unequal squares.¹⁶ His figure for dissecting the initial square later became a popular motif in the Islamic ornamental arts.

Abu'l-Wafa², apparently aware of the artisan's needs and his tendency to learn through perception rather than deductive reasoning, seems to use the method of dissection-and-composition deliberately. For the artisans to acquire a sound understanding of the basic principles of geometry, he may have thought, the most appropriate vehicle would be a method with which they were familiar.

When we examine the mathematical content of *Interlocking Figures*, we observe a parallel between its first seven folios — excluding an insert that concerns describing the complete circle of a given segment of a circle and another insert on a later folio — and Abu'l-Wafa²'s account of the *conversazioni* in Baghdad. The constructions in both these folios were based on the method of dissection-and-composition, again suggesting that the source of *Interlocking Figures* may be a series of *conversazioni*. The mathematicians who attended those gatherings seem to have shared Abu'l-Wafa²'s considerations, or perhaps followed a tradition which goes back to even earlier times. The constructions on the following ten folios, except two inserts of patterns, are also similar to certain chapters in Abu'l-Wafa²'s book.¹⁸ The topics covered in the first seventeen folios apparently constitute the basic principles and skills of geometry that were exemplified in *Geometric Constructions*, and may represent "teaching sessions" of *conversazioni* recorded in *Interlocking Figures*.

TEACHING SESSIONS

The common topic that ties together the constructions in the first seven folios is the application of the method of dissection-and-composition to regular figures. In most cases, the objective seems to be to dissect a regular figure — a triangle, hexagon, hexagram, octagon, or octagonal star — in such a way that the parts can be composed in a square. The purpose of these exercises was apparently to find the side of a square the area of which is equal to a given figure, that is to determine the area of a given figure in terms of a unit square. The insert on completing the circles (Euclid, *Elements* 3.25), which was introduced as a *ṣaḥl* (section), gives the impression that it was added when the discussion was diverted by a ques-

tion that was raised. The fact that a number of constructions were introduced into the discussion as a *ṣaḥl* (question) adds support to the idea that these discussions took place at *conversazioni*.

Two sets of constructions are of particular significance. The first, on folio 1, shows two congruent regular decagons dissected in such a way that when composed together with the pentagram that fills in the space between the five decagons touching one another, they form a larger decagon the side of which is equal to the radius of the circle that circumscribes the given decagon or the pentagram.¹⁹ The objective of this novel construction was apparently to prove through the method of dissection-and-composition that the radius of the circle that circumscribes the larger decagon is equal to the square of the side of the given decagon. In other words, the square of "extreme and mean ratio" (popularly known today as the "golden mean" or "golden section") is equal to one plus the extreme and mean ratio or to two plus the reciprocal of the extreme and mean ratio (that is, $1.618^2 \dots = 1 + 1.618 \dots = 2 + 0.618 \dots$). In this construction, if the given decagon is regarded as a unit, then the pentagram corresponds to the reciprocal of the extreme and mean ratio, and the larger decagon becomes two units plus the reciprocal. Although no mention is made of Euclid, *Elements* 13.9, extreme and mean ratio, and this property of extreme and mean ratio in the explanatory text, this construction is a clear indication that the mathematicians of the Islamic world occupied themselves with the properties of extreme and mean ratio. The silence on these points, however, suggests that they may have considered these points as too fine for an artisan to need to know. Abu'l-Wafa² apparently shared this opinion as he does not mention this ratio in his constructions of the regular pentagon and decagon, though it is cited several times in the proofs added by the scholiast.²⁰

The second construction of interest is actually a proof of Euclid, *Elements* 2.14 (or 6.13) through the method of dissection-and-composition.²¹ It is equivalent to the extraction of the square root. If the longer side of a given rectangle is extended by an amount equal to the shorter side and a semicircle is drawn on it, then the extension of the shorter side that meets the semicircle corresponds to the side of the square the area of which is equal to the area of the given rectangle. For proof, the rectangle is dissected into four parts which are then fitted into the constructed square. The obvious similarity between this construction and Abu'l-Wafa²'s proof of the Pythagorean theorem (Euclid, *Elements* 1.47) proposed at a *conversa-*

zione strongly suggests that this method was generally employed by mathematicians in teaching the basic principles of geometry to artisans.

These two constructions were clearly produced by mathematicians. In contrast to their elegance, a set of constructions on a folio towards the end and the one on the second folio, both also performed by the method of dissection-and-composition, are approximate and display an uncalled-for complexity.²² The figures were dissected into so many pieces that the whole point of the method is missed: the dissection should have been made in such a way that once the result is known the side of the required square could have been determined on the original figure. These constructions could hardly have served a didactic purpose. One has the impression that they were meant rather for amusement, like a jigsaw puzzle. They are not likely to have been produced by a mathematician, but rather by an artisan who fancied the method as a game. The incompatibility of these two types of constructions recalls the artisans' false constructions and Abu'l-Wafa's ingenious solution to the problem of "composing a square from three squares," and suggests that not all the constructions in *Interlocking Figures* were produced by the mathematicians; some were made by the artisans.

The second topic, which concerns angular measurements, was introduced by the lemma (*muqaddima*): "If a triangle is inscribed in a circle, the ratio of its angles is equal to that of the segments of the circle that they subtend."²³ In relation to the first two drawings the necessary conditions concerning the angles of the regular heptagon and nonagon were explained without showing how the constructions were actually done. The angle subtending the side of the heptagon was taken as the modular unit, and the other angles were determined as corresponding to two and four units. The construction of the heptagon occupied quite a number of mathematicians of the Islamic world from the tenth to thirteenth centuries, and this condition served as the basis for several solutions proposed by various authors. All the solutions were based on intersecting conic sections, except the approximate construction of Abu'l-Wafa.²⁴

For the nonagon, a table of seven sets of modular angles is given. These sets constitute all the possible combinations of angles for the construction of the regular nonagon; the table recalls a similar set of angles for the heptagon that Ibn al-Haytham (965–1041) had explained.²⁵ The construction of the nonagon necessitates the trisection of an angle, namely the acute sextuple angle, which corresponds to a cubic equation, like

the construction of the heptagon. The trisection of an angle occupied many mathematicians, both Greek and medieval.²⁶ Several solutions had been proposed, some of which were based on conic sections and some on the reduction of the problem to a mechanical solution that was called "verging" by the Greeks and "moving geometry" by the mathematicians of the Islamic world. Pappus of Alexandria (fl. ca. A.D. 320) tells us that verging procedures, performed by moving a ruler around a fixed point until, by trial and error, the required intercept was found to be equal to a given length, were very practical.²⁷ Considering this moving geometry as the most practical for ordinary purposes, such as when artisans needed to find the dimensions of geometric constructions that involved cubic equations, Abu'l-Wafa² apparently offered mechanical solutions to the problems of the trisection of an angle and the duplication of a cube or a sphere.²⁸

The mathematician's silence concerning the constructions of the heptagon and nonagon seems to suggest that he considered the conic sections to be too advanced a topic for artisans to deal with (and most mathematicians would agree). He may also have thought that the approximate construction of the heptagon and the verging constructions of the nonagon that had been offered by earlier mathematicians, such as Abu'l-Wafa², were also not practical enough for executing these figures in ornamental patterns, an idea that conflicts particularly with the approximate constructions of the pentagon and the patterns based on cubic equations that were treated in later folios of *Interlocking Figures*. It may be that these different groups of constructions were treated by different mathematicians. The one who explained the necessary conditions of the heptagon and nonagon appears to assume that the constructions would ordinarily be executed by using an instrument to set the required angles, like some of the patterns in the later folios.

The last construction concerning the topic of angular measurements explains a method of constructing the sextuple and quintuple angles.²⁹ According to this method, the sextuple angle corresponds rightly to 30° , but the tangent of the quintuple angle is equivalent to $\sqrt{3} - 1$, which corresponds to $36^\circ 12' 21'' 41'''$, an approximation of 36° . The combination of the sextuple and quintuple angles had, since Ptolemy of Alexandria (fl. A.D. 125–141), been used by astronomers to prepare the sine tables through the difference between them. The fact that an angular module of 6° was used in this construction points to that common practice in astronomy, and thus suggests that it may be an astronomer who dealt with this construction. It also recalls the method which

Ibn al-Razzaz al-Jazari (fl. 1206) used in setting the angular scale on an instrument similar to a protractor he made to set out any desired angle.³⁰ He determined the sextuple angle by aligning the alidade and the ruler along the sides of an equilateral triangle, but he did not explain how he marked the line to determine the quintuple angle. It would be farfetched to suggest that the construction in *Interlocking Figures* was perhaps related to the undefined quintuple angle of al-Jazari's instrument; nevertheless, it seems likely that it was associated with a protractor of some sort.

After two constructions of triangles concerning $\sqrt{5}$ and an insert, the last topic of the teaching sessions — the regular pentagon — was dealt with.³¹ It consists of four geometric constructions, each of which is identified by the opening of the compass, and two smaller figures, apparently added later, with no explanatory texts. Two of the constructions were performed by setting the opening to a given length that would correspond to the perpendicular of the required pentagon, one by setting it to the side, and the other to the diagonal (that is, the diameter of the circumscribing circle), which was proposed by Abu Bakr al-Khalil al-Tajiri. All the constructions were approximate.

Abu'l-Wafa² also offered several constructions for the regular pentagon in *Geometric Constructions*.³² They were all neat and simple constructions which were based on various propositions of Euclid and Ptolemy concerning extreme and mean ratio, the pentagon, and decagon. A correct construction of the regular pentagon can be produced only by utilizing extreme and mean ratio. The constructions in *Interlocking Figures*, on the other hand, do not reveal any use of this essential ratio (except perhaps the added figures: in one the diagonal of the half square is shown, the other is a pentagram drawn in a semicircle). We thus do not expect that the author of the elegant method of dissection-and-composition concerning the decagon was also the author of any of these constructions. They are generally lengthier than Abu'l-Wafa²'s constructions. Since each additional step means an additional possible source of inaccuracy in geometric constructions in general, they are also less practical. The fact that Abu Bakr al-Khalil was the author of one of the constructions appears to indicate the presence of mathematicians. What could be the reason for them, then, to propose these approximate constructions of the pentagon when other methods that were not only correct but also simpler and more practical were available in Abu'l-Wafa²'s book?

The answer may lie in the fact that all the pentagon

constructions in *Interlocking Figures* were performed by setting the compass opening to a certain length and not changing it throughout the process. Abu'l-Wafa² employed the same technique in a total of 18 constructions, some of which were offered as alternative solutions concerning the pentagon, in *Geometric Constructions*. We learn from al-Sijzi (fl. 963–990) that it was customary for artisans to perform geometric constructions by using a fixed compass opening. This technique, which was sometimes called “rusty compass”, can be traced back to the ancient Greeks, or perhaps to the ancient Indians.³³ We can thus deduce that Abu'l-Wafa² was trying to accommodate this long-lasting custom of artisans. The approximate constructions of the pentagon too seem to reflect the same practice, and some of them might have been produced by the artisans themselves. The reason why this technique was so popular among artisans, it seems, was its practicality. According to Abu'l-Wafa², to control the movement of the legs of a compass was a difficult task;³⁴ if they could be fixed for the whole process, however, the chances of error were considerably diminished.

Two inserts of ornamental patterns in the teaching sessions appear to be the precursors of the change of character that we observe in the remaining part of *Interlocking Figures*. The one preceding the pentagon constructions was introduced as a *mithāl* (example), and, interestingly enough, it proved to be an alternative construction to the one following the last pentagon construction. Thereafter, it appears, the main emphasis of the sessions of *conversazioni* shifted from the teaching function to working out various methods of constructing certain patterns and to designing new ones. These sessions, which I describe as “working sessions”, do not seem to serve a didactic purpose in relation to the basic principles and skills of geometry but to address some immediate problems concerning the artisans' practice.

WORKING SESSIONS

The later part of *Interlocking Figures* is even more disorganized than the beginning. As examples of the same or similar patterns do not generally follow one another in sequence, the principle behind the grouping is harder to identify than the topics of the so-called teaching sessions. It appears as a collection of ornamental patterns put together in a haphazard manner which seems to reflect the spontaneity of the problems related to their execution, and in some cases to their design. Two sets of constructions grouped around two patterns can still be identified through their mathematical content. These

patterns deserve attention since both were based on cubic equations, one of the most advanced topics of medieval mathematics. They thus manifest themselves as the irrefutable indicators of the contribution mathematicians made to Islamic art and architecture.³⁵ In so doing, they become important evidence to support the hypothesis that *Interlocking Figures* was the record of a series of *conversazioni* between artisans and mathematicians.

The treatment of these two patterns was also similar. One of the constructions in each set followed the definition of the problem and was performed through a verging procedure by the aid of the same instrument, the *gunyā miṣṭar* (T-ruler), and the other constructions were all based on approximate methods. To understand these verging constructions one first needs to understand what a T-ruler is. *Interlocking Figures* provides a detailed description of it:

The principle regarding the [preceding] pattern (*ʿaqd*) belongs to conics (*makhruṭat*). These can be drawn with the aid of an instrument called the T-ruler (*gunyā miṣṭar*). It is an instrument with which several patterns of conical shapes can be drawn. That is what this scribe (*kātib*) infers, but whether it is actually true is not known.

Be that as it may, one makes such a ruler in the manner of the alidad of an astrolabe (*ʿidāda-i asturlab*). In the middle of it, a perpendicular ruler is erected similar to the arrow (*sahm*) of the alidad of the “boat astrolabe” (*asturlab-i zawraqi*). It is called the “control of the mast” (*saṭāra-i gunyā*). As an example, the ruler ABGD is made of the [horizontal] ruler AB and the perpendicular [leg] GD [fig. 1]. As the edge AB of the ruler deviates from [the horizontal position], the edge GD deviates [together] from the vertical [position], like the deviation of a tailored (*mujayyab*) alidad. This deviation from the vertical line GD is distinguished on the face of the astrolabe alidad in the text. Point D would be fixed on the edge of the alidad so that the right angle GDA is found to be absolutely correct. It is perhaps possible that through this ruler several extraordinary proportions (*nisbathā-i gharīb*) can be created.³⁶

If the comment of the *kātib* is taken by itself, it is difficult to judge whether it was added by the copyist of the manuscript or by the compiler of *Interlocking Patterns*. His admission of ignorance of verging procedures is also reflected throughout the passage. He appears to confuse the verging procedures that were the mechanical equivalents to the solutions that involve conic sections with the drawings of the conic sections themselves and to doubt that such a complicated task could have been achieved by an instrument as simple as the T-ruler. The comment reflects the confusion displayed in the whole passage and particularly the remark at the end, and thus

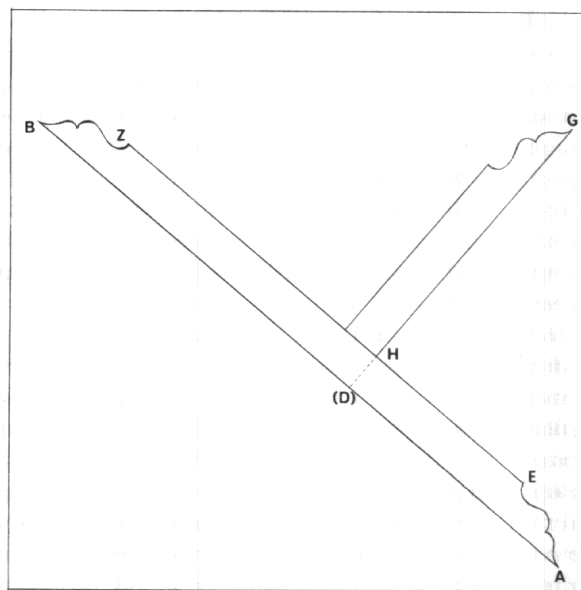


Fig. 1. T-ruler (adapted from MS Persan 169, sec. 24, fol. 191v).

suggests that the “*kātib*” referred to was more likely the compiler himself, who could hardly be a mathematician, than the copyist or the translator.

The T-ruler was apparently an instrument similar in principle to the T-square of today. It is difficult to ascertain, however, whether it was as common in those days. Was it an ordinary instrument that any artisan would have used to draw perpendicular lines, or was it a specialized instrument used only to deal with conic sections, as the description in *Interlocking Figures* suggests?³⁷ The comparison of the T-ruler with the alidad of an astrolabe implies that the ruler was the less common instrument of the two.

The alidad of an astrolabe is a flat ruler with sharpened ends, extending across the whole diameter of the back of the astrolabe, and rotating around the axis at its center.³⁸ Ordinarily, the two arms of the alidad form a straight line without a perpendicular leg. The anonymous author specified that the T-ruler was similar to the alidad of a “boat astrolabe”. Such a perpendicular attachment, therefore, must have been peculiar to that particular type of astrolabe. Among surviving examples, only two can be identified as “boat astrolabes”; but, unfortunately, in both cases the alidads are missing.³⁹ The “boat astrolabe” of al-Sijzi was a theoretical example of this type.⁴⁰ It has to be added, however, that even an ordinary astrolabe from the Islamic world, dated to the twelfth century, included a sort of perpendicular attach-

ment to the alidat and that two Flemish astrolabes from the sixteenth century also had T-shaped alidads.⁴¹

Abu'l-Wafa's *Geometric Constructions* and his account of a particular *conversazione* provide us with useful information to compare with the mathematical content of the so-called teaching sessions of *Interlocking Figures*. Omar Khayyam's untitled treatise appears to be the pertinent source to consult for the so-called working sessions in general and for one of the two ornamental patterns under consideration in particular. There he tells us that a question raised at a meeting motivated him to write the treatise. Through the remarks he made while analyzing the problem, we understand that that meeting was actually a *conversazione* between the mathematicians and artisans. He reduced the problem to constructing a right triangle in such a way that the sum of the perpendicular and the shorter side is equal to the hypotenuse. Taking this triangle as the basis, he reached the cubic equation

$$x^3 + 200x = 20x^2 + 2,000.$$

He then worked out the solution of the cubic equation by means of conic sections — the intersection of an hyperbola and a circle — but he also added, "For people who do not know conics, certain instruments are used."⁴² Several mathematicians attempted to devise certain instruments to draw conic sections, but these were theoretical efforts bearing no practical value.⁴³ When a problem of practical consequence involved conic sections, as we learn particularly from Abu'l-Wafa², a verging construction was usually carried out with the aid of a ruler. We can thus deduce that what Omar Khayyam advised artisans to use must have been a simple instrument like a ruler, i.e. the T-ruler.

One of the ornamental patterns of cubic equations in *Interlocking Figures* was recorded as follows:

The proportion of this pattern too is derived from conics. Its objective consists in constructing a right triangle in such a way that the sum of the perpendicular and the shorter side is equal to the hypotenuse. Ibn Haytham wrote a treatise on the construction of such a triangle, and in this work he described a hyperbola and a parabola (*qata'a-i makhru'at zayid wa mukafi*).

Here, however, the objective can be achieved with the aid of a T-ruler (*mis'ara gunya*) [fig. 2]. As mentioned in the preceding lemma, the objective of the pattern consists of four figures: pinecones (*sanaubari*) of two right angles surrounding an equilateral right-angled quadrilateral (*murabba'a*). For example, the pinecone quadrilaterals AIHK, GHMN, DMLS, and BLKO surround the square (*murabba'a*) KHML. Since the corner H of the square and of both figures is a right angle, then necessarily, the lines K[G and] HD are straight. Thus the triangle AKG is right

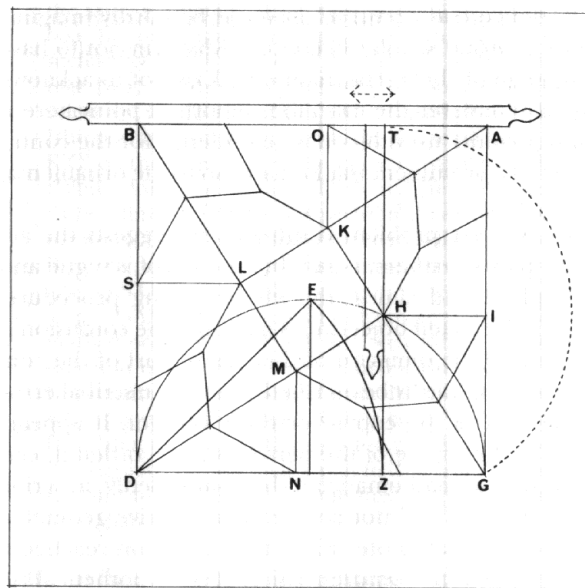


Fig. 2. Verging construction of the pattern based on Omar Khayyam's triangle (adapted from MS Persan 169, sec. 24, fol. 191r).

angled and congruent to the triangle GHD. Since the latter is a right triangle, it is inscribed in the semicircle [GED]. Hence, point H has to take place on the arc GE AE in the text.

[In order to perform the construction of the required pattern, place the T-ruler arbitrarily on the side AB of the given square.] As the corner T of the ruler is a right angle and the side AB is a straight line, [the perpendicular leg of the ruler cuts the arc GE at an arbitrary point H. If the line TH is extended to point Z on the side GD, then TH + HZ is always equal to the side AG,] which corresponds to the side AB of the square and the ruler. [Since it is also equal to the side GD, then TH + HZ = GD. Slide the T-ruler along the side AB and compare the segments TH and HG until a position of point H is reached at which TH = HG. Then HG + HZ = DG. Thus, the required condition is fulfilled. Mark this position of point H and complete the pattern accordingly.] Allah knows best.⁴⁴

The perfect agreement between the definition of the triangle that was discovered by Omar Khayyam and the lemma for this construction leaves no room to doubt that the triangle in question in both cases was the same. However, it is attributed here to Ibn al-Haytham. Was it a misquotation on the part of the scribe, or was the same triangle discovered by Ibn al-Haytham in a treatise which is now lost?⁴⁵ This triangle does not appear to be a general problem, such as the trisection of an angle or the construction of the regular heptagon, that would attract the attention of two mathematicians. Even if this unlikely

coincidence had occurred, it would be hardly imaginable for a proud scholar like Omar Khayyam not to have knowledge of Ibn al-Haytham's work or not to acknowledge it. Whatever the case, the important point here is that it provides us with concrete evidence for the contribution of a prominent mathematician to the ornamental arts.⁴⁶

The way the problem is introduced suggests the authorship of a mathematician; but the proof is vague and incomplete, and almost the whole verging procedure, which is the stated objective, is missing. The confusion it reflects and the omission of the crucial part of the construction can hardly be ascribed to a simple scribal error on the part of the copyist or the translator. It appears rather as the failure of the person who recorded it, certainly not a mathematician but supposedly a scribe whose training had not covered constructive geometry, in taking accurate notes when the discussion reached a point beyond his comprehension. The hypothetical reconstruction of the missing part implies a mathematician who was competent in cubic equations, and who presumably explained it at a session of the *conversazioni*.

In the untitled treatise, Omar Khayyam offered an additional approximate solution to the cubic equation through the tables of sines. He was apparently accommodating the practical needs of the artisans. In *Interlocking Figures* too, in addition to the verging construction, four approximate methods of geometric constructions were proposed for the same pattern. The deviations they yield range from 0.2 percent to 2.9 percent.⁴⁷

The problem on which the other ornamental pattern of cubic equations was based is explained in *Interlocking Patterns* as follows:

The construction of this pattern [is performed] by a T-ruler (*gunyā miṣṭar*). I say that in this pattern, that is the frame (*khāna*) ABD[G], the orange (*turanj*) ABZT is required to be congruent to the orange DGEH so that BZ will be equal to GE, and the [segment] EZ will be common to both [fig. 3]. Then BE will be equal to GZ. On the other hand, [by definition,] in the orange EGDH the sides EG, GD and EH, HD are equal to one another. Then, necessarily, the angle [G]E[H] will be equal to the angle D and be a right angle, and EZ is equal to EH. As this lemma is known now, let us assume that the side GD of the frame of the pattern is given. [The problem can then be reduced to determining] the unknown extremity of the side DB.⁴⁸

The detailed proof of the lemma for the required pattern allows us to outline the essentials of the problem. In any given right triangle BAG, when the bisecting line BT is drawn and the perpendicular TZ is erected on the hy-

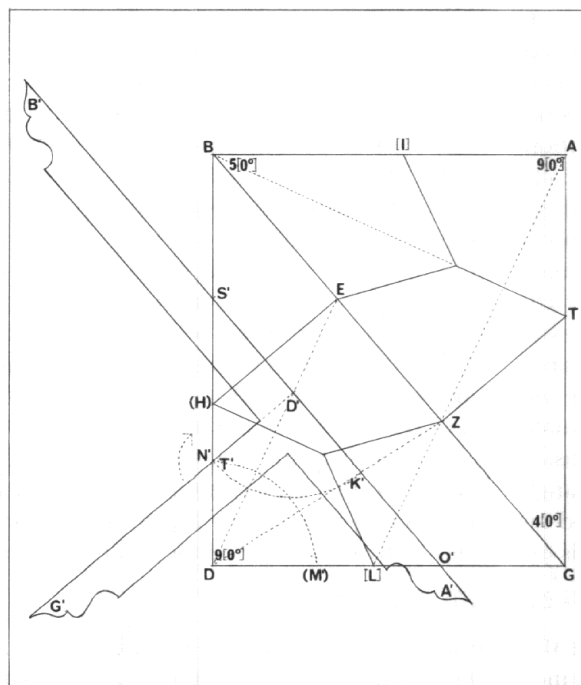


Fig. 3. Verging construction of the pattern in question (adapted from MS Persan 169, sec. 24, fol. 190r).

potenuse BG, necessarily AT is equal to TZ and AB is equal to BZ (fig. 3). Then ATZB becomes the special quadrilateral that has two pairs of equal sides, the opposite ones being unequal.⁴⁹ These general properties constitute the first condition of the problem. If point E is marked on the hypotenuse BG so that the segment ZE is equal to ZT, the required triangle comes into existence when the angle ABG is so determined that the remaining segments EB and ZG on either side of ZE become equal to each other. This specific requirement, then, is the second condition of the problem. If two such triangles are joined together along their hypotenuses to form a rectangular frame, and if the special quadrilaterals are subdivided into smaller ones having EZ as the common side, then the whole configuration becomes a neat interlocking pattern. In order to achieve this objective, the two conditions have to be met simultaneously through a cubic equation.

The fact that the same figure, the special quadrilateral, was referred to as "orange" in this construction and as "pinecone" in the previous one suggests that the two were not proposed by the same author. Both were apparently competent mathematicians. The author of this construction, if we are allowed to speculate, was perhaps

more skilled than the other in explaining such advanced topics to laymen as the supposed scribe's accurate recording here suggests. The author of this construction had probably worked out the cubic equation for the problem, as Omar Khayyam did in his untitled treatise.

The following is a hypothetical reconstruction of the procedure followed by this mathematician to reach the required cubic equation:

[Let us suppose that the required triangle BGD is constructed on the given line GD. Then $ZE = EH = HD = DL$, $BE = GZ = GK$, and $GD = GE = BZ$. The area of the triangle BGD is equal to the sum of its components, that is, $\frac{1}{2}(BG \cdot GD) = \frac{1}{2}(BE \cdot HE) + \frac{1}{2}(HE \cdot GE) + \frac{1}{2}(HD \cdot GD)$. But $HE = HD$ and $GE = GD$. Then $BD \cdot GD = BE \cdot HE + 2HE \cdot GD$. If every term of the equation is divided by GD^2 , then $BD:GD = BE \cdot HE:GD^2 + 2HE:GD$. Since the triangles BEH and BGD are similar, $BE:HE = BD:GD$. Then $BE:HE = BE \cdot HE:GD^2 + 2HE:GD$. But $GL = GD - DL$ and $BE = GL$ and $EH = DL$. Then $BE = GD - HE$. When BE is substituted by $(GD - HE)$ and the equation is simplified, it becomes $GD^3 - HE \cdot GD^2 = 3GD \cdot HE^2 - HE^3$. Therefore, since GD is the given length, HE has to be determined in order to perform the construction. Consequently, if GD is designated as the unit (1) and HE as the unknown (x), then the problem is reduced to one of solving the algebraic equation

$$x^3 + 1 = 3x^2 + x.]$$

When this cubic equation is solved by modern means, the theoretical value of the angle BGD can be obtained with precision: $49^\circ 28' 53'' 35'''$.

According to Omar Khayyam's classification of polynomials up to the third degree in his book, *Al-jabr w'al-muqābala* (Algebra), this particular equation is an example of the third type of tetranomial equations.⁵⁰ He demonstrated that the solution to this type could be obtained by the intersection of two hyperbolas. The author of this construction, for his own satisfaction, might have solved this equation accordingly. For the sake of the artisans, however, he was expected to offer a practical verging construction. The verging procedure following the lemma in *Interlocking Figures* was, in all probability, proposed by this mathematician in order to realize the pattern in question:

[To start the construction, we draw the given line GD and erect the perpendicular DB on point D (fig. 3).] We then take the [T]-ruler [and mark point D' on the edge of the ruler as the intersecting point of the two perpendicular edges].⁵¹ With an arbitrary [compass] opening [*fath* that is, radius], we mark the segments D'K', D'T', equidistant from point D'. We then [go back] to the frame of the [required] pattern [and] mark [equidistant] points M', N' from the corner D on the sides DG, DB, in accordance with

the same segment that we have marked on the ruler. We take the ruler [again] and put [it upon the triangle GDB in such a way that] the letter T' on the edge of the perpendicular [leg] is positioned precisely on point N'. Then we move the ruler around this point from left to right until [the position is reached at which] the segments S'D' and K'O' on the edge of the ruler become equal to each other. [While doing this,] point T' should never be separated from point N'. [When this position is reached,] we draw the line S'O'. [To determine the required position of point B,] from point G, which is known, we draw the line GB parallel to S'O' [so that it becomes the diagonal] of the rectangle [GABD. By drawing lines from point D through the marked positions of D' and K'], we divide the line BG in the ratio of S'D':D'K':K'O' [which divide the line S'O']. Concerning the ruler which is parallel to GB, the segment S'D' is equal to K'O', and K'D' is equal to D'T' [and to] DT', [that is, DN'.] With respect to the line GB, therefore, the segment BE becomes equal to ZG, and ZE is equal to EH and to HD. These [properties] constitute the objective. Allah knows best.⁵²

In the figure of the original manuscript, the T-ruler is placed upon the triangle GAB. The explanation in the text, however, makes sense only if the ruler is placed upon the triangle GDB. This mistake, however small, was one not likely to have been made by the copyist whose task would only have been accurately to reproduce what was in front of him. It seems instead to result from the confusion of the supposed scribe when dealing with matters of which he admitted having no knowledge. What we observe here, aside from some minor confusion, is a full account of a verging construction and thus concrete evidence for the use of such procedures in Islamic art.

The problem not only serves as the basis for a neat interlocking pattern, but also becomes the source for a set of interrelated proportions. In the triangle ABD (fig. 3), point I divides the shorter side into two parts so that the ratio between them is equal to the ratio between the longer and shorter sides: $BI:IA = AG:AB$ {1}.⁵³ Point T divides the longer side into two parts so that the ratio between them is equal to the ratio between the hypotenuse and the shorter side: $GT:TA = GB:AB$ {2}. Points E and Z divide the hypotenuse into three parts so that the ratios between them are equal to ratio {1}; and the ratios between the hypotenuse and the combinations of two parts are equal to ratio {2}: $BE:EZ = GZ:EZ = AG:AB$; $GB:BZ = GB:GE = GB:AB$.

In addition to the verging procedure, four approximate methods were offered for constructing this pattern. The most indelicate of these was recorded in a marginal note on the same page as the verging construction, in contrast to the attention paid by the mathematician to accuracy:

However, [concerning this pattern,] we have found a construction by approximation (*taqrīb*). If we divide the right angle into 9 equal parts, four *sic* parts constitute the angle AGB, and 5 parts constitute the angle [A]BG; but it is a substantial approximation [fig. 3].⁵⁴

We may infer from the fact that this method was added later that the scribe and some others, presumably artisans, who had attended the particular *conversazione* measured the angle ABG afterwards on the constructed pattern, and found that it was approximately 50° . We can also infer that they had at their disposal an instrument for taking angular measurements. As they observed, this represents a considerable deviation of 1.9 percent from the theoretical value. This approximate method seems to confirm Abu'l-Wafa's view that for artisans, in contrast to mathematicians, the theoretical correctness of a geometric construction was not of prime concern.⁵⁵

Another approximate method was presented in two constructions. The explanation given in the text concerned the first one (fig. 4), which was designated as a "proportion of convenience" (*nisbat-i sahl*):

[Let a perpendicular be erected on point A of a given line AB.] As the corner A is a right angle and AB is a side of it, the construction is to be made so that the corner A AB in the text is bisected by the line AD. The quarter-circle BDG is drawn so as to determine [the position of] point D. Then, taking point B as the center and BA as the [compass] opening, the arch AH is drawn. When [the compass opening] BD [describes] the arc DE, [it] becomes concentric with the arc AH. Then point E is made the center and when [the compass opening] EB [describes] the arc BHZ, [it] intersects the arc AH at point H. If the line ZHT is drawn [tangent to the arc AH at point H], then the segment GT becomes equal to ZH. This is the proportion of the triangle that is required in the [following] pattern [fig. 5], which is drawn [accordingly]. Allah knows best.⁵⁶

The gist of this construction (fig. 4) is as follows: if BD is taken as the side of an octagon inscribed in a circle the radius of which is AB, then the cosine of the required angle ATZ is equal to $AB:2BD$. In this method the required angle corresponds to $49^\circ 12' 37'' 54'''$ and yields a deviation of 1 percent. The pattern is drawn using the procedure of the preceding construction (fig. 5).

Another approximation method was recorded in a note added between the two preceding constructions:

Some of the artisans draw this pattern in such a way that the length [of the frame] is divided into seven parts and the width into six parts. This is a very close [approximation]. Allah knows best.⁵⁷

Indeed, this numerical method proves to be remarkably

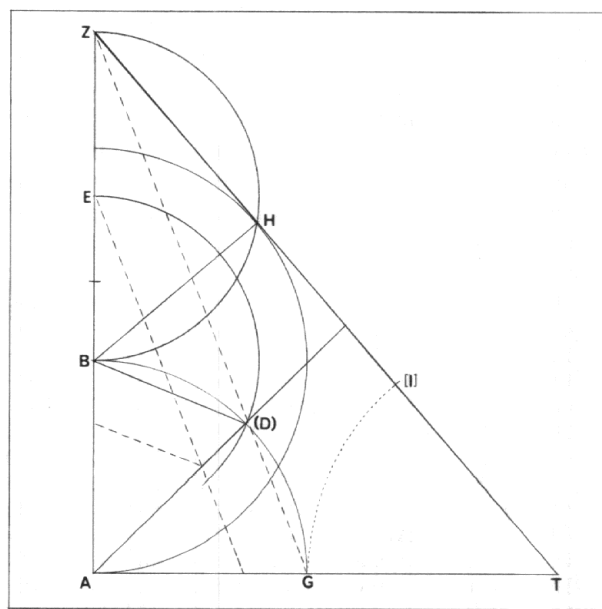


Fig. 4. An approximate construction of the triangle of the pattern in question (adapted from MS Persan 169, sec. 24, fol. 187v).

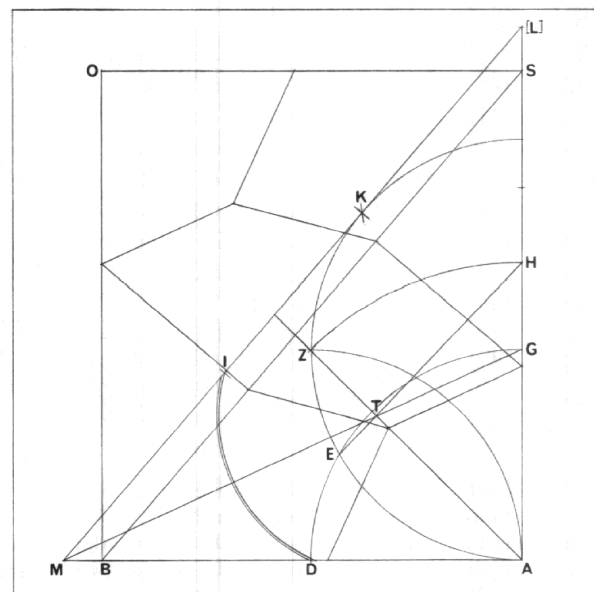


Fig. 5. An approximate construction of the pattern in question based on the triangle in fig. 4 (adapted from MS Persan 169, sec. 24, fol. 187v).

accurate when compared with others, contrary to the general belief that geometric methods are superior to the numerical ones. The ratio 7:6 corresponds to the tangent of the angle $49^{\circ} 23' 55'' 20'''$; and the deviation it yields is no more than 0.3 percent. This note is significant in two respects: it provides concrete evidence for the collaboration of artisans with mathematicians in creating ornamental patterns, and it points to the fact that numerical ratios were common enough tools for artisans in dealing with geometry.⁵⁸

The last approximate method of construction concerning the pattern in question was designated as an "example" (*mithāl*):

Let us suppose that the square AGED is constructed on the half of the [given] side AB, and the line GET is extended (fig. 6). With the [compass] opening AB the quarter[circles] BTZ [and ATH] are drawn so that point T is determined on the perpendicular TG. The lines TA and TZ are drawn so that TA intersects the line ED at point I. Then, [by taking the compass] opening TI from point Z, the segment ZK is cut off [on the line ZT]. When the line KA [intersects] the quarter-circle ATH, [the position of] point L is determined. This [point] constitutes the requirement.⁵⁹

The procedure of this construction is based on the property of the pattern that requires the extension of the transverse diagonal AL of the "orange" ABLD to pass

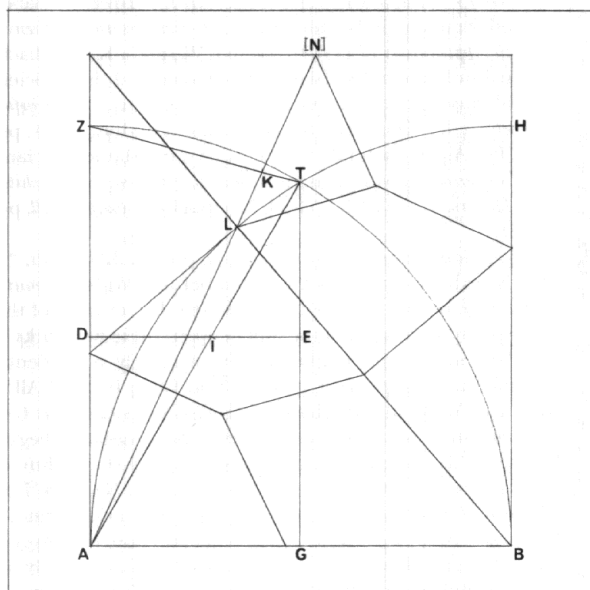


Fig. 6. An approximate construction of the pattern in question (adapted from MS Persan 169, sec. 24, fol. 185v).

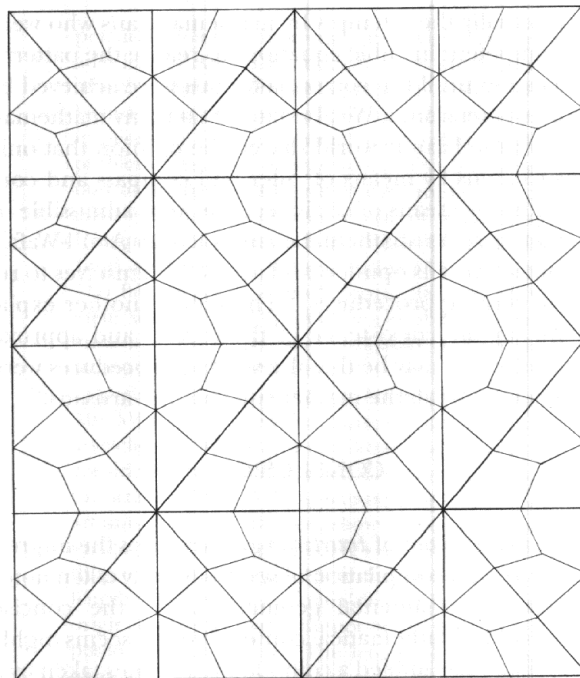


Fig. 7. The composition generated by repeating the pattern in question.

through point N.⁶⁰ It thus reaches its objective through approximation, with fairly accurate results: the angle ABL corresponds to $49^{\circ} 15' 9'' 31'''$ which produces a deviation of only 0.8 percent.

All these approximation methods would have been accurate enough had the pattern been used as a single ornamental motif. Except for the angular one, they would still be satisfactory if the compositions were generated using only a few repetitions. However, when the task was to execute compositions of numerous interlocking patterns on large surfaces — which was usually the case in the ornamental arts related to architecture — the accumulation of inherent inaccuracies using approximation methods would have been intolerable.⁶¹ Under such circumstances, the correct solution — the verging construction in this particular case — would answer the need (fig. 7).

The numerical approximation method evidently and the angular one probably were proposed by artisans, but the geometric ones give the impression of being offered by mathematicians. In *Interlocking Figures* the verging constructions of this pattern and the one based on Omar Khayyam's triangle were the last of their groups. We may thus speculate that most of the approximate methods

were actually the attempts by mathematicians who were not competent in cubic equations to realize the patterns in question until the correct solutions were achieved by the competent ones. We also know that many mathematicians of the Islamic world shared the opinion that only constructions by means of ruler and compass and constructions by means of conic sections were admissible in geometry; some mathematicians, such as Abu'l-Wafa², did not share this opinion and allowed themselves to resort to verging procedures.⁶² Therefore, another explanation for the coexistence of the correct and approximate solutions may be that the verging procedures were not rejected by all the mathematicians and artisans.

CONCLUSION

The general layout of *Interlocking Figures* gives the impression of being a compilation of spontaneously taken notes concerning ornamental geometry. After the concise analysis of its mathematical content here, it seems highly likely that it was indeed a compilation of notes taken by a scribe at a series of *conversazioni* between artisans and mathematicians. References to artisans and to mathematicians, particularly the citation of Abu Bakr al-Khalil, indicate the presence and contributions of these two groups of professionals. The incompatibility among various constructions, ranging from game-like ones to the ones that involve cubic equations, reflects this duality in *Interlocking Figures* and suggests that it can hardly be the work of a single author. The compiler, whose knowledge does not seem to tally with the advanced knowledge involved in certain constructions, appears to be a scribe, as he is referred to in the text. If our conclusion is true, then what we observe in *Interlocking Figures* is the documentation of a strong collaboration between mathematicians and artisans, a phenomenon that sheds new light particularly on the formative period of Islamic art and architecture.

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NOTES

Author's note: I would like to express my gratitude to Zaka Siddiqi who translated into English the passages quoted from the Persian text. Letters designating points in the geometrical figures are transcribed as follows:

alif = A, bā³ = B, jīm = G, dāl = D, hā³ = E, zay³ = Z, hā³ = H, tā³ = T, yā³ = I, kāf = K, lām = L, mīm = M, nūn = N, sīn = S, ʿayn = O.

When letters are mentioned in the text but are missing on the figures, they are added in parentheses. My own additions to the figures are between brackets or distinguished as dotted lines.

1. It was Necipoğlu who first pointed to this link in her comprehensive study of the available sources: "Its importance lies in documenting an intimate link between the type of practical geometry popularized in late-tenth-century Baghdad by such mathematicians as al-Buzjani and the later scroll tradition" (Gülru Necipoğlu, *The Topkapı Scroll: Geometry and Ornament in Islamic Architecture* [Santa Monica, Calif., 1995], p. 138).
2. It is described in E. Blochet, *Catalogue des manuscrits persans de la Bibliothèque Nationale* (Paris, 1905–1934), 2: no. 772, 41–47. Some of the works in the same manuscript are also described in C.A. Storey, *Persian Literature*, 2 vols. (London, 1972), 2:3, 9, 25, 27–30, 52, 54. The manuscript is thought to have been copied in the early seventeenth century. *Interlocking Figures* has been translated into Russian by Midhad S. Bulatov in "Vvedenie [v uchenie] o podibnikh i sootvetstvennikh figurakh", appendix to *Geometricheskaia garmonizatsiia v arkhitekture Srednei Azii IX–XV vv.*, 1st ed. (Moscow, 1978), pp. 325–54. Since then it has attracted the attention of quite a few historians of Islamic art and architecture; see *ibid.*, pp. 41–61; *idem* "U istokov arkhitekturnoy nauki Srednego Vostoka," *Narody Azii i Afriki* 1 (1973): 92–101, 252; Renata Holod, "Defining an Art of Architecture," in *Architectural Education in Islamic World*, ed. Ahmet Evin (Geneva and Singapore, 1986), pp. 26–32; *idem*, "Text, Plan and Building: On the Transmission of Architectural Knowledge," in *Theories and Principles of Design in the Architecture of Islamic Societies*, ed. Margaret B. Ševčenko (Cambridge, Mass., 1988), pp. 1–12; Lisa Golombek and Donald Wilber, *The Timurid Architecture of Iran and Turan* (Princeton, N.J., 1988), pp. 137–73; Wasma'a K. Chorbachi, "In the Tower of Babel: Beyond Symmetry in Islamic Design," *Computers and Mathematics with Applications* 17 (1989): 751–89; Wasma'a K. Chorbachi and Arthur L. Loeb, "An Islamic Pentagonal Seal (From Scientific Manuscripts of the Geometry of Design)," in *Fivefold Symmetry*, ed. I. Hargittai (Singapore and London, 1992), pp. 283–305; Alpay Özdural, "Omar Khayyam, Mathematicians, and *Conversazione* with Artisans," *Journal of the Society of Architectural Historians* 54 (1995): 54–71; Necipoğlu, *Topkapı Scroll*, pp. 131–81.
3. Bulatov, *Geometricheskaia garmonizatsiia*, pp. 52, 325; *idem*, "U istokov arkhitekturnoy," p. 99; Golombek and Wilber, *Timurid Architecture*, p. 159. Necipoğlu questioned the certainty of this assumption by saying that the link between these two works in the Paris manuscript might have been merely coincidental (Necipoğlu, *Topkapı Scroll*, p. 169). Abu Ishaq b. ʿAbd Allah Kubanani Yazdi, the translator of Abu'l-Wafa's *Geometric Constructions*, informs us that he completed the translation begun by one of his contemporaries, Shaykh Najm al-Din Mahmud, who undertook it at the request of Amir Mansur Nasr ibn ʿIraq (Bulatov, *Geometricheskaia garmonizatsiia*, pp. 51–52; François Woepcke, "Analyse et extrait d'un recueil de constructions géométriques par Aboul Wafā," *Journal Asiatique* 5, no. 5 [1855]: 358–59). Bulatov identified the latter as Abu Nasr Mansur b. ʿAli ibn ʿIraq (d. 1018–36), and thus dated this translation to the early eleventh century. Necipoğlu accepted the eleventh-century translation as original and dated Kubanani Yazdi's copy

- to the Timurid period by mentioning that he was active in 1442–43 and in 1459 (Necipoğlu, *Topkapi Scroll*, p. 168).
1. Bulatov, "U istokov arkhitekturnoy, pp. 99, 252; idem, *Geometricheskaia garmonizatsiia*, p. 52.
 5. Chorbachi, "In the Tower of Babel," p. 755.
 6. Necipoğlu, *Topkapi Scroll*, p. 169.
 7. Özduval, "Omar Khayyam and the Artisans," p. 64. In a forthcoming publication I plan to translate the complete document and study its contents using information gathered from other works on mathematics, particularly *Geometric Constructions*, and other drawn material on ornamental geometry, particularly design scrolls, for comparative purposes. This article is a preliminary assessment of the mathematical content of *Interlocking Figures*, highlighting certain points of interest, particularly the use of cubic equations in the ornamental arts.
 8. MS Ayasofya 2753, Süleymaniye Kütüphanesi, Istanbul. For other manuscript copies and modern publications of Abu'l-Wafa's work, see Özduval, "Omar Khayyam and the Artisans," pp. 67–68. For a brief description and analysis of this work, see Necipoğlu, *Topkapi Scroll*, p. 138.
 9. Ca'fer Efendi, *Risāle-i Mi'mariyye: An Early-Seventeenth-Century Ottoman Treatise on Architecture*, trans. and ed. Howard Crane (Leiden, 1987), pp. 22–23, 28.
 10. See Necipoğlu, *Topkapi Scroll*, pp. 157–58.
 11. Apart from *Geometric Constructions*, which is exclusively for the use of artisans, Abu'l-Wafa wrote another work for scribes and other officials, *Kitāb al-manāzil fīmā yahtāju ilayhi al-kuttāb wa al-ʿummāl wa ghāiruhum min ʿilm al-ḥsāb* (Book of the stations [chapters] on what scribes, officials and other require of the science of arithmetic). It includes chapters on the application of geometry to mensuration, arithmetical operations, and their applications in certain fields. For a Russian translation of this work, see M.I. Medevoy, "Ob arifmeticheskoy traktate Abu'l-Wafy," *Istoriko-matematicheskie issledovaniya*, 13 (1960): 253–324; for an Arabic edition, see A.S. Saidan, *Arabic Arithmetic: The Arithmetic of Abu'l-Wafa* (Amman, 1971); for an English summary, see idem, "The Arithmetic of Abu'l-Wafa," *Isis* 65 (1974): 367–75.
 12. Webster's defines a *conversazione* as "a meeting, reception, or assembly for conversation and social recreation or for discussions of art, literature, or science" (*Webster's Third New International Dictionary*, ed. Philip B. Gove [Springfield, Mass., 1986], p. 498); the Oxford definition includes the following: "from about the close of the 18th c., chiefly applied to assemblies of an intellectual character, in connexion with literature, art, or science" (*Oxford English Dictionary*, 12 vols. [Oxford, 1970], 2: 941).
 13. For a more detailed discussion on these references and on *conversazioni*, see Özduval, "Omar Khayyam and Mathematicians," pp. 54–63. For Omar Khayyam's treatise, see Ali R. Amir-Mo'ez, "A Paper of Omar Khayyam," *Scripta Mathematica* 26 (1963): 323–37; G.H. Muṣāhib, *Omar Khayyam as an Algebraist* (Teheran, 1960); S.A. Krasnova and B.A. Rosenfeld, "Omar Khayyam pervy algebraicheskiy trakta," *Istoriko-matematicheskie issledovaniya* 1963, fasc. 15.
 14. MS Persan 169, sec. 24, fols. 189r, 187v, 188v.
 15. Necipoğlu's opinion is essentially the same as mine, but differs on the following points: "The collaboration of this unknown professional mathematician [Abu Bakr al-Khalil] with practitioners recalls the example of al-Buzjani, who held meetings with artisans in tenth-century Baghdad where problems of practical geometry concerning their works were discussed. In addition to Abu Bakr al-Khalil, the anonymous author of *A'māl wa ashkāl* [*Interlocking Figures*] — who seems to have been a *muhandis* with practical rather than theoretical training in geometry — also provided some solutions of his own to problems raised in these sessions. The haphazard organization of the treatise suggests that its anonymous author compiled it as a spontaneous record of ongoing experiments (Necipoğlu, *Topkapi Scroll*, pp. 169–70). For the citation of al-Tajiri, see MS Persan 169, sec. 24, fol. 187r. For the two works of Abu Bakr al-Khalil, see MS Persan 169, sec. 16, fols. 108–18; idem, sec. 19, fols. 124v–138. For a brief description of the contents of these works, see Necipoğlu, *Topkapi Scroll*, p. 168; Storey, *Persian Literature*, 2: 25.
 16. Ayasofya 2753, fols. 47, 53–60. Since all the constructions in Abu'l-Wafa's book involve constructive geometry, some scholars believe that he had no recourse to arithmetic or to metric considerations, and that it was the only mode of construction for artisans. Abu'l-Wafa's demonstration of the falsity of the artisans' methods, however, shows that he did occasionally resort to such considerations. He said: "We understand that [in this construction] the sides of the [required] square have a length which is equal to one side plus half a diagonal [of the original squares]; but it cannot be the side of a square composed of three squares. The length of the latter is actually more than the former. If we make the side of each square approximately 10 cubits [*dhira*], those who are trained know that the side of the required square should be equal to $17 + \frac{1}{2}$ cubits by approximation; whereas, in this construction the side of the [required] square is equal to $17 + \frac{1}{2} - \frac{1}{7}$ [cubits]. There is a considerable difference between the two. Also, when the [original] square BG is bisected [into two] and each half is placed on either side of the square, then [the line] BG does not correspond to the line HK or TI for two reasons. Firstly, the diagonal of the square BG is incommensurable; whereas the line HK and the side of the square BG plus the other half are commensurable. [Secondly,] due to that, the diagonal of the square BG is $14 + \frac{1}{2}$ [cubits] by approximation, but the side HK is 15 [cubits]. It is therefore proven that this dissection-and-composition [of a square] is false" (Ayasofya 2753, fols. 54–55). We observe that Abu'l-Wafa expressed the approximate values of $\sqrt{3}$ and $\sqrt{2}$ in terms of principal fractions by assigning a length of 10 cubits to the side of the unit square. It is of interest to note, firstly, that in *The Stations*, his book on practical arithmetic for scribes and other officials, he explained in great detail how to use this sort of fraction, which he apparently proposed for their elegance (see above no. 11). He did not explain how these values were calculated, but his remark, "who are trained know by approximation," suggests that in his time a traditional method of approximation was in use among artisans and/or mathematicians. According to Youschkevitch, the mathematicians of the Islamic world always had the tendency to express incommensurable ratios in terms of approximate commensurable ones, and they gradually revived and extensively used a second definition of proportion which was known by the Greeks before Eudoxus (Adolf P. Youschkevitch, *Les Mathématiques arabes*, trans. M. Cazenave and K. Jauoishé [Paris, 1976], p. 84). As Kennedy notes, this early definition of proportion made use of the procedure for expressing a ratio as a continued fraction (E.S. Kennedy, "The Exact Sciences of Iran under the Saljuks and Mongols," *Cam-*

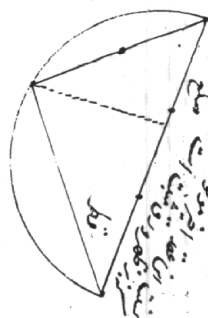
- bridge *History of Iran*, ed. J.A. Boyle [Cambridge, 1968], 5: 663). If the numerical values of $\sqrt{3}$ and $\sqrt{2}$ are calculated by means of continued fractions, the values cited by Abu'l-Wafa², $17 + \frac{1}{3}$ and $14 + \frac{1}{7}$, correspond respectively to the second and fifth cuts of these procedures. If expressed in terms of simple fractions, they correspond to $26/13$ and $99/70$. The procedure of continued fractions renders a set of numerical approximation series of such irrational magnitudes which can be traced back to the ancient Mesopotamian and Hindu mathematics; see Otto Neugebauer, *The Exact Sciences of Antiquity* (New York, 1969), pp. 146–48.
17. MS Persan 169, sec. 24, fols. 180r–183r.
 18. Ibid., fols. 183v–187r.
 19. Ibid., fol. 180r.
 20. Ayasofya 2753, fols. 16–17, 20, 23–27.
 21. MS Persan 169, sec. 24, fol. 182v. For Euclid's propositions, see Euclid, *The Thirteen Books of Elements*, trans. Thomas L. Heath, 3 vols. (New York, 1956), 1: 409–10, 2: 216.
 22. MS Persan 169, sec. 24, fols. 180v, 197r.
 23. Ibid., fols. 183v–184v.
 24. For the solutions based on conic sections, see Jan P. Hogendijk, "Greek and Arabic Constructions of the Regular Heptagon," *Archives of the History of Exact Sciences* 30 (1984): 197–330. Abu'l-Wafa² offered two constructions, one on a given side and one inscribed in a circle, for the regular heptagon (Ayasofya 2753, fols. 18, 25). In both cases, the side of the heptagon was determined as being equivalent to half of the side of the equilateral triangle inscribed in the same circle. These two were the only approximate constructions in his book, which he probably borrowed from Hero of Alexandria (fl. 1st century A.D.) who expressed the same relation in terms of a numerical ratio, $7/8$; see Hero of Alexandria, *Codex Constantinopolitanus*, ed. and trans. E.M. Bruins, 3 vols. (Leiden, 1964), 3: 230.
 25. For Ibn al-Haytham's explanation, see R. Rashed, "La construction de l'heptagone régulier par Ibn-al-Haytham," *Journal for the History of Arabic Science* 3 (1979): 309–87. See also, Hogendijk, "Regular Heptagon," 236–37.
 26. See Jan P. Hogendijk, *On the Trisection of an Angle and the Construction of a Regular Nonagon by Means of Conic Sections in Medieval Islamic Geometry* (Utrecht), Dept. of Mathematics, preprint 113. See also idem, "How Trisections of the Angle Were Transmitted from Greek to Islamic Geometry," *Historia Mathematica* 8 (1981): 417–38; Wilbur Knorr, "On the Transmission of Geometry from Greek into Arabic," *Historia Mathematica* 19 (1983): 71–78.
 27. Thomas Heath, *A History of Greek Mathematics*, 2 vols. (Oxford, 1965), 1: 239.
 28. Ayasofya 2753, fols. 13–14. Most of the Greek works, which included the verging solutions to these problems, were already translated into Arabic by the ninth century and were therefore available to Abu'l-Wafa². His two constructions of the trisection of the angle were based on the ones solved by Archimedes (*Book of Lemmata*, 8) and Pappus (*Collections* 4.36–42), and his construction of the duplication of the cube was borrowed from Hero (*Mechanics* 2.11). It can also be argued that Abu'l-Wafa² would have offered a verging solution to the construction of the regular heptagon, instead of the approximate one he proposed, had there existed any workable one (the solution attributed to Archimedes had a certain resemblance to verging procedures, but actually it was not workable). For other verging constructions in Greek mathematics, see Heath, *Greek Mathematics*, 1: 235–41, 255–68; idem, *The Works of Archimedes* (New York, 1921), pp. c–cxxii; Maximilian Curtze, *Reliquiae Copernicanae* (Leipzig, 1875), pp. 7–21.
 29. Ms 169, sec. 24, fol. 184v. Since the terms *gunyā* 6 (sextuple right-angled) and *gunyā* 5 (quintuple right-angled) are used in the explanatory text, it is generally considered as evidence for the existence of the specially prepared set squares that were used by artisans in their works (Bulatov, *Geometricheskaia garmonizatsiia*, p. 332; Alpay Özduval, "An Analysis of the Geometry of Stalactites: Buruciye Medrese in Sivas," *Middle East Technical University Journal of the Faculty of Architecture* 11 [1991]: 61; Necipoğlu, *Topkapı Scroll*, p. 170). The operative word here is *gunya* (Greek: corner, angle; joiner's square). Abu'l-Wafa² treated *kunya* in the introductory chapter on the instruments employed in geometric constructions. He described several geometric methods to construct a right-angled corner and to check the correctness of the already made ones, and added that the artisans preferred other methods in checking the accuracy of these instruments, and explained how they utilized the two rational right triangles, namely 3–4–5 and 5–12–13, to that effect (Ayasofya 2753, fols. 5–7). There was no mention of the lengths of arms to indicate any special set square. While describing a construction of the regular pentagon, he referred to the isosceles triangle having each of the angles at the base double the remaining one (36°) as "the triangle of the pentagon" (*al-muthallath al-mukhāmmas*), and added that it is used in other constructions (idem, fols. 16–17). It was a clear reference to *Euclid* 4.10, which was the initial proposition concerning the regular pentagon. *Gunya* 5 thus appears to be a version of the triangle of the pentagon, a geometrical figure, rather than to represent a real set square. It can be argued also that had *gunya* 5 been an instrument, it would have simplified the construction of the regular pentagon, to the point where the whole exercise concerning the pentagon in *Interlocking Figures* would be rendered futile.
 30. Ibn al-Razzāz al-Jazarī, *The Book of Knowledge of Ingenious Mechanical Devices*, trans. Donald R. Hill (Dordrecht and Boston, 1974), pp. 196–98.
 31. MS Persan 169, sec. 24, fols. 186r–187r.
 32. Ayasofya 2753, fols. 16–17, 23–24.
 33. To ridicule the incorrect construction of the regular heptagon by Abu'l-Jud (fl. 969–990), al-Sijzi wrote: "But as regards his lemmata and similar lemmata: finding the hexagon in the circle was difficult — this is what the carpenters construct on the tops of pots with one (fixed) opening — not to speak of finding the heptagon" (Hogendijk, "Construction of the Heptagon," p. 311). Al-Sijzi's reference to the carpenters, though inappropriate, implies that constructions with fixed compass openings is what they ordinarily do. In the Arabic translation of Pappus's *Collections* 8, which dealt with the instruments and machines for various crafts, a section was devoted to constructions with fixed openings, sometimes called a "rusty compass" (J.L. Berggren, *Episodes in the Mathematics of Medieval Islam* [New York, 1986], p. 90). It is interesting to note that a manuscript of this translation was copied by al-Sijzi. According to Youschkevitch, such constructions can be traced back to the "rules of cords" of the ancient Indians; but Abu'l-Wafa² was the first mathematician who treated them systematically (Youschkevitch, *Mathématiques arabes*, p. 108). In medieval Europe too that sort of application appears to be a common practice. Mathes Roriczer, a fifteenth-century master mason, illus-

- trated an approximate pentagon construction with a fixed compass opening in the booklet *Geometria deutsch*; and Albrecht Dürer repeated the same construction (Lon R. Shelby, *Gothic Design Techniques* [London and Amsterdam, 1977], pp. 116–17, 180–81). The practicality of fixed compass openings aroused the interest of Leonardo da Vinci and several Renaissance mathematicians, and such constructions were then widely circulated in Europe (Youschkevitch, *Mathématiques arabes*, p. 108).
34. Ayasofya 2753, fol. 4.
 35. Necipoğlu expressed a similar view on this issue: "The many Arabic treatises on conic sections often deal with problems of practical application. Their contribution to architectural practice and to the decorative arts awaits assessment by historians of science so that the ways in which theory and praxis interacted can be understood more clearly" (Necipoğlu, *Topkapı Scroll*, p. 141).
 36. MS Persan 169, sec. 24, fol. 191v; see appendix.
 37. We know that several geometers, such as al-Kuhi (fl. 970), Ibn al-Husayn (fl. 1200), al-Sijzi, and Ibn al-Haytham, attempted to devise an instrument, which they called "the perfect compass" (*al-birkār al-tāmm*) or "the conic compass" (*birkār al-qutūʿ*), to construct conic sections in practice; see M. François Woepcke, "Trois traités arabes sur le compas parfait," in *Notices et extraits des manuscrits de la Bibliothèque Impériale et autres bibliothèques* 22 (1874): 1–175; Jan P. Hogendijk, *Ibn al-Haytham's "Completion of Conics"* (New York, 1985), pp. 35–36, 61. It can be defined generally as a specialized compass: one arm, of fixed length, could be fixed at an arbitrary angle to the plane of the paper, while the second arm, of variable length, could be revolved at another given angle around the first arm; the second arm then described a right cone, so its endpoint described a conic section on the plane of the paper. It is doubtful whether the perfect compass had any real practical value since, in all the known manuscripts, conics are usually drawn as arcs of circles. For other types of drawing instruments, see Eilhard Wiedemann, "Über geometrische Instrumente bei den muslimischen Völkern," in *Gesammelte Schriften zur Arabisch-Islamischen Wissenschaftsgeschichte*, vol. 1, *Schriften 1876–1912* (Frankfurt am Main, 1984), pp. 417–33. Verging constructions were ordinarily carried out by means of rulers. There are references in the Greek literature to certain mechanical devices that would be used in special cases; see Heath, *Greek Mathematics*, I: 238–40. There is no such reference, as far as we know, in Islamic literature, except a wishful remark by Banu Musa (fl. ninth century): "It is possible, when this [kind of] device has been found, for us to divide any angle we wish into three equal divisions" (Marshall Clagett, *Archimedes in the Middle Ages*, 5 vols. [Madison, Wisc., 1964], I: 345).
 38. For more detailed information on astrolabes, see Robert Gunther, *The Astrolabes of the World*, 2 vols. (London, 1976); Willy Hartner, "The Principle and Use of the Astrolabe," *A Survey of Persian Art*, ed. Arthur U. Pope and Phyllis Ackerman, 15 vols. (Tehran, 1977), 6: 2530–54; idem, "Asturlab," *Encyclopedia of Islam*, 2nd ed.; I: 722–28.
 39. Both of these examples are from India; Gunther, *The Astrolabes*, pp. 218–20.
 40. Ahmad al-Sijzi, *ʿAmal al-Asturlāb*, Topkapı Sarayı Ahmet III Kütüphanesi (Istanbul), 3342/9.
 41. Gunther, *The Astrolabes*, pp. 280–81, 379, 391–92.
 42. Amir-Moéz, "A Paper," p. 329. For other sources on Omar Khayyam's treatise, see above n. 13. When this cubic equation is solved, one of the acute angles of Omar Khayyam's triangle corresponds to $57^{\circ} 03' 53'' 34'''$.
 43. See above no. 37.
 44. MS Persan 169, sec. 24, fol. 191r; see appendix. The figure and the translation here contain some corrections of the ones in Özdural, "Omar Khayyam and the Artisans," pp. 64–65 (which was based on Bulatov's Russian publication), though they do not alter the general character of the information there. Notably, the instrument used in the verging construction was precisely a T-ruler instead of an interpreted combination of a triangle and a ruler; both in fact worked according to the same principle. An additional point is that the anonymous author drew the arc EZ with a radius DE which cuts the side DG at point Z. This construction is not related to the required verging procedure. It seems that the author added another approximate method to determine point H without any explanation. Indeed, according to this method, $DG: \sqrt{2}$ is a very close approximation of the theoretical value of DZ. It is of interest to note also that in the figure of the original manuscript the main triangular subdivisions of the square pattern display a remarkable accuracy.
 45. None of Ibn al-Haytham's treatises was about the lemma quoted in *Interlocking Patterns*. Ibn al-Haytham's autobiography includes a list of 70 of his works written before 10 February 1027. In *ʿUyūn al-anbāʾ fi ṭabaqāt al-aʿyibāʾ* (Sources of information on the generations of physicians), Ibn Abi ʿUṣaybiʿa (d. 1270) adds a list, composed by Ibn al-Haytham and containing the titles of 21 treatises written between 10 February 1027 and 24 July 1028, and another list of 92 abbreviated titles of works which Ibn al-Haytham wrote before 2 October, 1038 — only five or six of these works can be found in earlier lists. Aside from a few of the extant works which are not mentioned in any of the lists, these seem to cover practically all the works of Ibn al-Haytham. For more information on the titles and contents of Ibn al-Haytham's works, see H.M. Said, ed., *Ibn al-Haytham, Proceedings of the Celebrations of the 1000th Anniversary* (Hamdard, 1971); A.I. Sabra, "Ibn al-Haytham," *Dictionary of Scientific Biography*, ed. C.C. Gillispie, 14 vols. (New York, 1972), 6: 189–210; Fuad Sezgin, *Geschichte der arabischen Schrifttums*, 7 vols. (Leiden, 1967–), vol. 5 (1974): 365–74; vol. 6 (1978): 254–61; Hogendijk, *Ibn al-Haytham's "Completion of Conics"*, pp. 52–62. Necipoğlu is of the opinion that Ibn al-Haytham might have dealt with this triangle, not in a separate treatise, but in the concluding chapters of *Kitāb al-abniya wa al-ʿuqūd* (Book of buildings and constructions [patterns?]), which is now lost (Necipoğlu, *Topkapı Scroll*, p. 178, n. 37). It has to be noted, in regard to the problem solved by Omar Khayyam, that if the given and unknown quantities are designated differently, another cubic equation, $x^3 + 2x^2 = 1$, can be obtained. The solution of this equation requires the intersection of a hyperbola and a parabola.
 46. The question whether the reference in *Interlocking Figures* was to Ibn al-Haytham or Omar Khayyam is important only in regard to its date, which still remains uncertain. Bulatov assumed that it was written as an appendix to Abu'l-Wafa's work, and thus dated both to the early eleventh century. He added that the patterns found in the document can be traced to the eleventh-to-twelfth-century monuments of Central Asia; Bulatov, *Geometricheskaiia garmonizatsiia*, pp. 51–52, 325. The same date was accepted by Golombek and Wilber. They sug-

- gested, however, that some of the drawings in the document may have been added during the Timurid period; Golombek and Wilber, *Timurid Architecture*, p. 159. Chorbachi and Loeb traced a pentagonal seal on the decorations of the Abbasid palace in Baghdad dating from 1180–1230, and suggested, based on the drawings of the pentagon in the document, that it may have been compiled before this date; Chorbachi and Loeb, "Pentagonal Seal," pp. 301–4. This is a rather tenuous argument. In addition to the uncertainty of the historical information, the pentagon and pentagram were known since ancient times. Relying on information based on the translation by Bulatov, which proved to be inconclusive after consulting the original manuscript, I suggested earlier that it was probably written in early-thirteenth-century Diyarbakir (Özdural, "Omar Khayyam and the Artisans," pp. 65, 70, n. 46). Necipoğlu suggested that the Paris manuscript, which is thought to have been copied in the early seventeenth century, originally may have been compiled in the Timurid period, since the latest works it contains were composed in the first half of the fifteenth century. She is of the opinion that it may have been put together for an architect-engineer. By possibly associating Abu Bakr al-Khalil with the unidentified Abu Bakr, whose treatise on applied mensuration had been translated into Latin as *Liber mensurationum* by Gerard of Cremona (1114–82), she thought the early eleventh century may not be an unlikely date for *Interlocking Figures*. She added, however, that it seems to have been compiled sometime between the early eleventh and early thirteenth centuries, and appears to have been revised while it was being copied in the Timurid period by Kubanani Yazdi (see above n. 3); Necipoğlu, *The Topkapı Scroll*, pp. 167–69, 180, n. 115. If the reference in *Interlocking Figures* was to Omar Khayyam's treatise, which I am inclined to believe, then it cannot possibly be compiled before the late eleventh century.
47. These are discussed in detail in Özdural, "Omar Khayyam and the Artisans," pp. 65–66.
 48. MS Persan 169, sec. 24, fol. 190r; see appendix.
 49. This figure, which was peculiar to Islamic art, does not correspond to a specific term in modern geometry. It was usually referred to as "almond" by the artisans of the Islamic world; Özdural, "Omar Khayyam and the Artisans," pp. 57, 69, n. 17.
 50. Daoud S. Kasir, *The Algebra of Omar Khayyam* (New York, 1931), p. 102.
 51. In the figure of the original manuscript, the points that belong to the T-ruler and those used to perform the verging procedure were distinguished by red ink. Some of these, however, were identical to the ones that were used for the pattern itself; and no differentiation was made in the text. In order to avoid the confusion they create, the letters designating the points of the ruler and of the verging procedure are distinguished here by adding primes, both in the figure and in the text.
 52. MS Persan 169, sec. 24 fol. 190r; see appendix.
 53. By two exchanges, $BI:AG = IA:AB$. But $BI = GL$ and $IA = AT$. Then $GL:AG = AT:AB$. Therefore, when the lines BT and AL are drawn, the triangle BAT is similar to the triangle AGL. Hence, the angle ABT is equal to the angle GAL. But, according to the general properties of the "orange", the transverse diagonal AZ is perpendicular to the longitudinal diagonal BT, and the angle ABT is equal to the angle TAZ. Consequently, when the transverse diagonal AZ is extended, it coincides with the line AL in the pattern in question.
 54. MS Persan 169, sec. 24, fol. 190r, top right-hand margin; see appendix.
 55. Abu'l-Wafa² tells us that artisans and surveyors were not at all concerned with diagrammatic proofs, but were satisfied with approximations of geometric constructions so long as these were perceived to be correct. The mathematicians believed in the correctness of a geometric construction only if it could be proved by deduction (Ayasofya 2753, fol. 52).
 56. MS Persan 169, sec. 24, fol. 187v; see appendix.
 57. Ibid., see appendix.
 58. See above n. 16.
 59. MS Persan 169, sec. 24, fol. 185v; see appendix.
 60. See above n. 53. In this construction, the approximation is achieved by taking ZT as the side of the duodecagon inscribed in a circle, the radius of which is AB, and marking point K on it so that the segment ZK is equal to AB ($1 - 1:\sqrt{3}$). Assuming that this approximation fulfills the necessary theoretical condition, the construction is completed by following the theoretical properties of the pattern: the line NKA cuts the arc AH at point L so that LB becomes equal to AB and thus the "orange" ABLD can be formed.
 61. Abu'l-Wafa² tells us that when artisans try to join together a large number of figures constructed by approximation methods, in spite of all their efforts, they can never accomplish the task. He advised them, in such circumstances, to follow the procedure of mathematicians explained in his book rather than the customary procedure of artisans (Ayasofya 2753, fol. 21). Concern for this problem appears to have been the main reason why Abu'l-Wafa² wrote his book on geometric constructions for artisans.
 62. For other verging constructions in Islamic mathematics, see Hogendijk, "Regular Heptagon," p. 201, n. 4; A.I. Sabra, "Ibn al-Haytham's Lemma for Solving 'Alhazen's Problem'," *Archives of the History of Exact Sciences* 26 (1982): 304–5, n. 10; Clagett, *Archimedes*, 1: 666–69.

APPENDIX

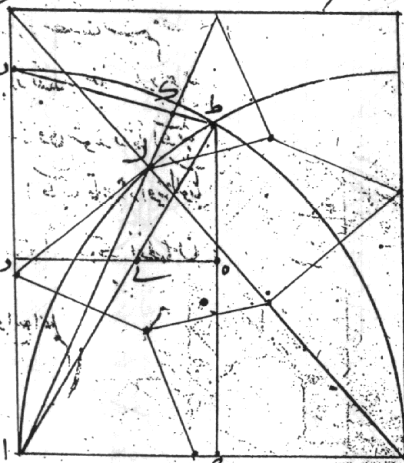
Fī tadakhul al-awhkal al-mutashabiha aw al-mutawafiqā, MS persan 169, fols. 185v, 187v, 190r, 191r, and 191v. Reproduced by permission of the Bibliothèque Nationale de France, Paris.

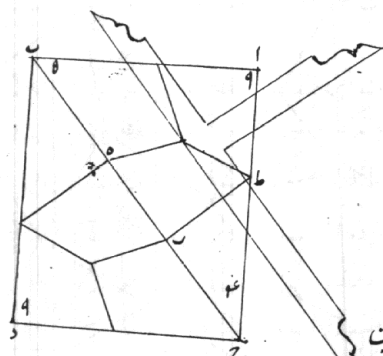


فصل این مقدار احتیاج علی
قام از اوید داریم که چون از او رشت
در طرف خلق اقطر فصل کنیم و از تمام فصل خودی اخراج
کنیم تا خلق او وسط اقطر کند و از منحنی قطع از او باقی بماند و عود
باشد و حصول این چنین شئی شکل است و از اصول اقلیدس خلق می
و آن در علم موجودات و عمل هر یک سطح حاصل می آید چون
مقدار عود و عرض باشد

برضا ضلع آیه عرضی مربع آخره و دل
کند و خط ح که با اخراج کند و بفسخ آیه
مربع ب ط را بگرداند تا بر عود ط نقطه ط
دریابد و خط ط ا ط را پیوندد تا ط آه و با
قطع کند نقطه ای که از نقطه و بفسخ
ط که بعد از رکت وصل کند که خط ک آ
بر محیط ربع ا ط ح نقطه آل در بدو این

مطلوب

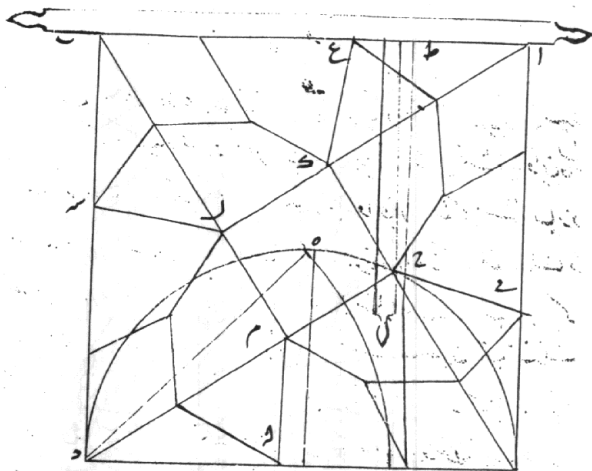




این مثلث با مثلث دیگر
برابر است و از این رو
که در این دو مثلث
زاویه ها برابر است
و اضلاع مقابل
به این زاویه ها
برابر است

علی بن ابی طالب میگوید که درین
عدد که در خانه است ای می باید که طریقی
چون چنین باشد که مثل سه بود و هشت
و دیگر چون طریقی که در خانه است
زاویه ها مثل زاویه ها بود و در این
فرض کنیم که از خانه ضلع هر معلوم است
بر داریم و از نقطه در سطح مقدار یک
بریم که همین مقدار که بر سطح نشان
بر ضلع و در آن سطح بر داریم و حرف
بر نقطه آنکه برین نقطه مسطره را از ج
سه و مثل که کرد و دو نقطه از نقطه
که معلوم است خط موازی می کشیم و
منت کشیم که چون بر سطح موازی
مثل خط است از خط مقدار سه و هر

اینست و الله اعلم



ثبت این عقیده بر آن محسوس است که در اینجا مقصود عمل تثلیث قائم الزاویه که عمود آن منش
 با ضلع افتر منضم مثل و نیز زاویه بود و این هضم در عمل این منش رساله گفته است و عمل او در
 منش و عمل محسوسات زاویه و شکافی منش اما اینجا ازین مسطح کوینا مقصود حاصل می توان کرد
 چنانکه گویم در عقیده اول مقصود این عقیده را چهار شکل است صوری قائم الزاویه بن محیط
 یک مربع متساوی الاضلاع قائم الزاویه مثل مربع صوری است که ح ح م و ح م ل
 باشد که محیط بر مربع که ح م ل اکنون چون زاویه ح در مربع و شکل دو قائم است
 بغیر از خط که ح مستقیم باشد و منش آن که ج قائم الزاویه بود و منش
 ح ح د باشد و این منش چون قائم است در نصف قوس واقع شود پس نقطه ح
 بر قوس آه طلب باید کرد پس هرگاه که ما را در مسطره زاویه ط قائم باشد و
 طرف آب مستقیم و منطبق بر ضلع آب از مسطح و مسطره را واسه

اعلم

حقیقت نسبت این عقد از مخروطانیت و به الی که آنرا کونیا منظر خوانیم توان کشیدن
 و آن الی باشد که بسیار عقود شکل مخروطانی بدین آن توان کشید و این مایه
 استقامت کاتب است که کرده است معلوم نیست و اینجانبان باشد که مسطره بازو
 مانند عضده اسطیلاب و بر وسط آن مسطره ای راست کند مانند سم عضده
 اسطیلاب زورقی و این را سطره کونیا خوانند مثل مسطره اب حرد که
 از وی مسطره اب بود و قائم حرد و باید که از مسطره طرف اب منحرف
 بود چون انحراف عضده ایجیب و سم جنب از قائم طرف حرد منحرف باشد
 و از قائم خط حرد بر روی عضده مین باشد و نقطه در طرف عضده مین
 و از او به حرد قائم عظیم درست باید که برین مسطره بسیار نسبتها
 غریب شکل بودن کرد اول در طریق عمل نسبت این عقد

