volts dropped very greatly in magnitude though it never completely disappeared. This critical potential has been previously observed by Jones and Whiddington (Phil. Mag. **6,** 889, 1928). The conditions under which they observed it were quite different from those existing in the present apparatus but the behavior of the peak was quite similar. They attributed it to an energy loss characteristic of the hydrogen molecule and this hypothesis seems to gain additional support from the present work. Unfortunately the most recent theoretical work on the hydrogen molecule contains no explanation for an energy loss of this value. However, it is not contrary to any experimental work as hydrogen has not previously been investigated under the conditions necessary to bring the effect into evidence.

Accurate data has been obtained for the angular distribution of scattered electrons with from one hundred to two hundred volts energy in both atomic and molecular hydrogen. The experimental points obtained for elastically scattered electrons in atomic hydrogen lie within the limits of experimental error on the theoretical curve predicted by the new quantum mechanics. This is of a different order of magnitude from the scattering which would be expected classically. The scattering in molecular hydrogen is somewhat greater than in atomic hydrogen. This is in general what would be expected from theoretical considerations. No accurate comparison with the theory is possible, however, for molecular hydrogen, as the scattering to be expected has not been calculated for this case.

The inelastic scattering in hydrogen has also yielded some very interesting results which will be discussed more fully in a forthcoming article. But one point should be mentioned here in as much as one statement made in the PHYSICAL REVIEW article previously referred to has been found to be in error. The decrease in the number of electrons scattered inelastically as one proceeds from small to large angles is more rapid than the similar decrease in the number of elastically scattered electrons. The reason for this is not clear, but the same thing has been observed by Dymond (Proc. Roy. Soc. **122,** 571, 1929) for electrons scattered in helium.

These results will be reported in full in an article to appear shortly in the PHYSICAL REVIEW.

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Palmer Physical Laboratory,
    June 15, 1929.

The Uncertainty Principle

The uncertainty principle is one of the most characteristic and important consequences of the new quantum mechanics. This principle, as formulated by Heisenberg for two conjugate quantum-mechanical variables, states that the accuracy with which two such variables can be measured simultaneously is subject to the restriction that the product of the uncertainties in the two measurements is at least of order h (Planck's constant). Condon* has remarked that an uncertainty relation of this type can not hold in the general case where the two variables under consideration are not conjugate, and has stressed the desirability of obtaining a general formulation of the principle. It is the purpose of the present letter to give such a general formulation, and to apply it in particular to the case of angular momentum.

* E. U. Condon "Remarks on Uncertainty Principles" Science LXIX, p. 573 (May 31, 1929), and in conversations with the writer on this topic.

We define the "mean value" A_0 of an (Hermitean) operator A in a system whose state is described by the (normal) function ψ as

$$A_0 = \int \bar{\psi} A \psi d\tau$$

where the integral is extended over the entire coordinate space. The Hermitean character of A (i.e.

$$\int \bar{\phi} A \psi d\tau = \int \bar{\psi} A \phi d\tau$$

for arbitrary ϕ, ψ) insures the reality of A_0. The "uncertainty" ΔA in the value of A is then defined, in accordance with statistical usage, as the root mean square of the deviation of A from this mean, i.e.

$$(\Delta A)^2 = \int \bar{\psi} (A - A_0)^2 \psi d\tau.$$

The uncertainty principle for two such variables A, B, whose commutator $AB - BA = hC/2\pi i$, is expressed by

$$\Delta A \cdot \Delta B \geq h \, |C_0| \, /4\pi$$

i.e. the product of the uncertainties in A, B is not less than half the absolute value of the mean of their commutator.

We here confine ourselves to sketching the proof of this principle for a one-particle system and for quantum mechanical variables $A(q, p)$, $B(q, p)$ which are linear in the momenta (p_x, p_y, p_z).[1] (The proof for the general case in which the operators can be expanded in powers of the momenta can be made along exactly the same lines.) Writing

$$A = a + a_x p_x + a_y p_y + a_z p_z$$

where $p_x = (h/2\pi i)\partial/\partial x$, etc. and the a's are functions of position, the Hermitean character of A requires that these functions be real and that div $(a_x, a_y, a_z) = 0$. The expression for $(\Delta A)^2$ may be written, on integrating once by parts, using the fact that div $(a) = 0$ and discarding the resulting surface integral, in the form

$$(\Delta A)^2 = \int |(A - A_0)\psi|^2 d\tau.$$

We are now in a position to apply the Schwarzian inequality[2]

$$\left[\int (f_1\bar{f}_1 + f_2\bar{f}_2)d\tau\right]\left[\int (g_1\bar{g}_1 + g_2\bar{g}_2)d\tau\right] \geqq \left|\int (f_1 g_1 + f_2 g_2)d\tau\right|^2$$

Taking

$$\bar{f}_1 = (A - A_0)\psi = f_2, \ g_1 = (B - B_0)\psi = -\bar{g}_2$$

and reducing the integral on the right hand side by integration by parts we find

$$\Delta A \cdot \Delta B \geqq \tfrac{1}{2}\left|\int \bar{\psi}(AB - BA)\psi d\tau\right|,$$

the required result.

[1] Cf. proof of special case $A = p$, $B = q$ in H. Weyl "Gruppentheorie und Quantenmechanik" pp. 66, 272.

[2] Weyl, l. c. p. 272.

We obviously obtain Heisenberg's result if the two variables are conjugate, for then C, and consequently C_0, are ± 1. As a further illustration of the principle, we apply it to the case of angular momentum. Here we have

$$M_x = yp_z - zp_y, \ M_x M_y - M_y M_x = -hM_z/2\pi i$$

so *the product of the uncertainties in two of the components of angular momentum is not less than $h/4\pi$ times the mean value of the third component in the state under consideration.* Consider in particular the state, treated by Condon, defined by

$$\psi = f(r)e^{im\phi}P_l{}^m(\cos \theta)$$

where the pole of the spherical coordinates lies on the z-axis. Then M_z, $M^2 (= M_x^2 + M_y^2 + M_z^2)$ have the definite values

$$M_z = M_{z0} = mh/2\pi, \ M^2 = l(l+1)(h/2\pi)^2$$

the mean values of M_x, M_y are zero and the uncertainties are given by

$$(\Delta M_x)^2 = (\Delta M_y)^2 = \tfrac{1}{2}[l(l+1) - m^2](h/2\pi)^2,$$
$$\Delta M_z = 0.$$

Now from the uncertainty principle for M_x, M_y we find

$$l(l+1) \geqq m(m+1)$$

which is in fact the case. This example shows that for $m = l$ the equality holds; the inequality is consequently the most restrictive one that can be deduced for angular momenta, for we have here a case in which the ultimate limit has (in principle) been reached.

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    Princeton, N. J.,
        June 18, 1929.

The Emission of Positive Ions from Metals

During an investigation of the critical potentials in metallic vapors the writer observed that considerable positive ion currents were obtained when the metals were heated. A study of these currents by positive ray analysis has yielded the following results. Fe, Ni, Cu, Zr, Pt, Pd give alkaline ions which persist for a considerable time when the metals are heated. The persistence in iron was most apparent, as a sample of electrolytic iron showed measurable potassium ion emission after having been heated for 200 hours in a vacuum of 10^{-8} mm and at a temperature of 850°C.

W, Mo, Ta, and Rh gave alkaline ions initially but these disappeared after a few minutes' heating. When the temperature of the metals was increased to the point where vaporization became appreciable, positive ion emissions were again observed. The atomic weights of these ions check within the limits of error with the atomic weights of the respective metals.

It is hoped to extend these results to other metals as well as to study the emission as a function of the temperature.

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    June 24, 1929.