The Geometry of Polygon-Shaped Buildings

Union and overlap, the Rub el Hizb star, and the semicircles that paid for the tower

Take two squares and rotate one 45°. The eight-pointed star you get is bigger than either square — but by exactly how much? A prime minister sketched that shape on paper, and an architect had to turn it into 88 floors of rentable office space.

GRADE BAND	TIME	FORMAT
9–10 · High School Geometry	**Two class periods**	**Video, dynamic geometry construction, composite area analysis**

Part 1 covers regular polygons, the hexagon construction, and which polygons tessellate. This part is composite area, and it stands alone if needed.

WHAT STUDENTS WILL BE ABLE TO DO

- Break a complicated figure into simple pieces and find its area by adding and subtracting.
- Use the union rule — $A + B -$ overlap — instead of measuring an awkward shape directly.
- Find the exact area of the eight-pointed star formed by two rotated squares.
- Calculate how much floor space a designer gains by adding semicircular lobes.
- Test a geometric model against a real building's published dimensions.

STANDARDS

7.G.B.6 area of objects composed of triangles, quadrilaterals, and polygons · **HSG-CO.A.3** symmetries of regular polygons · **HSG-C.B.5** areas of sectors · **HSG-GPE.B.7** perimeters and areas using coordinates · **HSG-MG.A.1** and **A.3** modeling with geometry and design problems

START HERE

Open the Voyager story in Chrome and look straight down on the Petronas Towers — that is the floor plan. Have students count the points, then notice the rounded bulges between them.

Then take a guess and write it on the board: *what fraction of that floor plate is star, and what fraction is bulges?* Classes reliably underestimate the bulges, which makes the §4 answer land.

earth.google.com/web/data=MkEKPwo9CiExNVVmU3Zpc24tX1pBbUtBZnFCUU0zeDRveG5PMUxrajASFgoUMDY2NDk5RTBERjEyOUNCOUEzNUQgAQ

01 The composite shape of the Petronas Towers

VIDEO **Petronas, Segment 1 — composite figures in the towers**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PetronasSegmentGoogleVoyagerSegment1.mp4

Designed by César Pelli, completed 1998, 451.9 m — the tallest buildings in the world until 2004 and still the tallest twin towers anywhere.

The floor plan comes from the **Rub el Hizb**, an Islamic symbol of two overlapping squares, one rotated 45°, forming an eight-pointed star. It appears throughout Islamic art and marks divisions of text in the Qur'an.

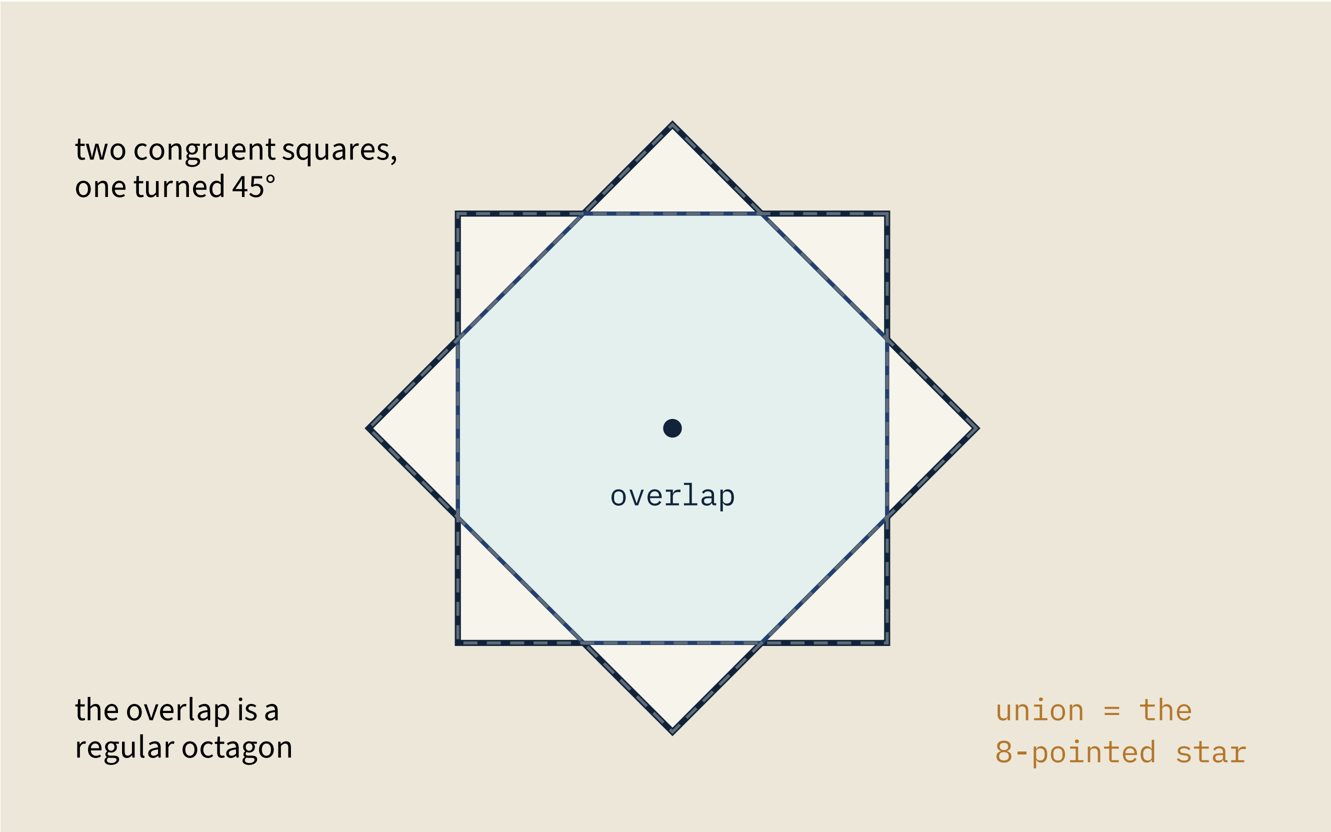
The design history is unusually well documented and worth telling. Pelli's early proposals were rejected for not looking Malaysian enough; Prime Minister Mahathir Mohamad sketched the interlocking-squares plan himself. Pelli then hit a practical problem — *the star did not enclose enough floor area to be worth building* — and solved it by adding a semicircle at each of the eight inner angles. That is the composite figure this lesson takes apart, and the fact that it exists for a *money* reason is the hook.

Composite figure	A shape built from simpler shapes joined, overlapped, or cut away.
Union	Everything covered by either shape: $A + B -$ overlap.
Intersection	The region covered by both at once — the overlap.
Octagram	An eight-pointed star polygon; the Rub el Hizb is the one made from two squares.
Floor plate	The area of one floor. Multiply by floors for total floor space.

02 Creating the composite shape

VIDEO **Petronas, Segment 2 — constructing the star on the TI-Nspire**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PetronasSegmentGoogleVoyagerSegment2.mp4

Draw a square, copy it, rotate the copy 45° about the same center. Everything below follows from those two squares.



Two squares, one octagon of overlap, eight points sticking out.

03 Analysis of the composite shape

VIDEO **Petronas, Segment 3 — area of the composite shape**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PetronasSegmentGoogleVoyagerSegment3.mp4

Slicing the star's sixteen sides into triangles works but is painful. The short route is the single most useful idea in composite-area work:

$$\text{area of union} = A + B - \text{overlap}$$

The squares are known, so the only real work is the overlap — which for two congruent squares rotated 45° about a shared center is a **regular octagon**.

Every side of both squares lies $s/2$ from the center, so the octagon has apothem $s/2$:

$$\text{octagon} = 8(\sqrt{2} - 1)m^2 = 2(\sqrt{2} - 1)s^2 \approx 0.8284\ s^2$$

$$\text{star} = 2s^2 - 2(\sqrt{2} - 1)s^2 = \mathbf{(4 - 2\sqrt{2})s^2 \approx 1.1716\ s^2}$$

The star is only about **17% larger** than a single square — a much smaller gain than the dramatic silhouette suggests, and precisely the problem Pelli ran into. Let students predict this before computing; almost everyone guesses far higher.

A CLEAN SURPRISE

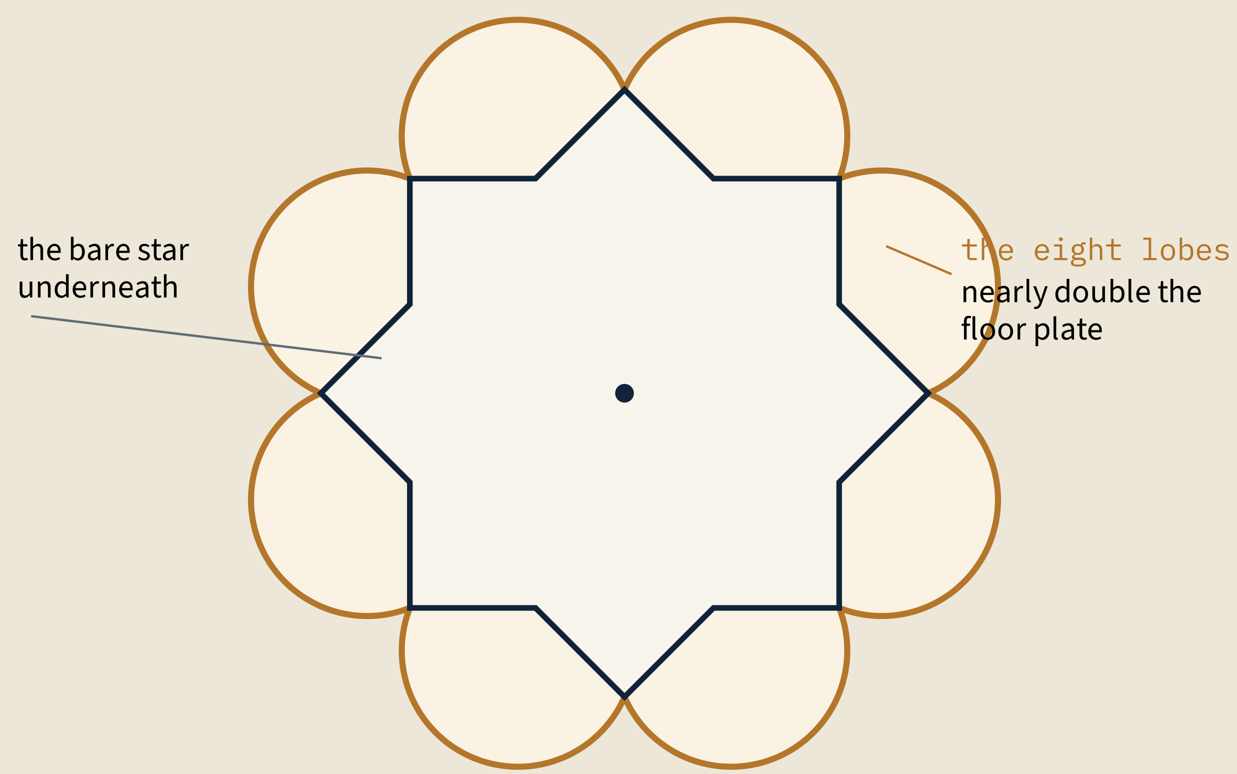
$$(4 - 2\sqrt{2})/(2\sqrt{2} - 2) = \sqrt{2}$$

The star is *exactly* √2 times its own overlap octagon — no rounding — and √2 is the same number that entered when the square was rotated. Have students multiply top and bottom by (√2 + 1) and watch it collapse. A rare case where a tidy exact answer falls out of an apparently messy figure.

04 Analysis of the enhanced composite shape

VIDEO **Petronas, Segment 4 — adding the circular area**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PetronasSegmentGoogleVoyagerSegment4.mp4

Between each pair of neighbouring star points is a notch. Span it with a semicircle bulging outward and the plan gains a rounded lobe. Eight times.



Star in cream, lobes in gold. The bulges carry about as much area as the star itself.

Take the semicircle's diameter to be the line joining two neighbouring star points, which sit $R = s/\sqrt{2}$ from the center, 45° apart:

$$\text{chord} = 2R \sin 22.5° \approx 0.5412\ s \ \rightarrow\ \text{lobe radius} \approx 0.2706\ s$$

PIECE	AREA	SHARE OF THE PLATE
The eight-pointed star	1.1716 s^2	50.2%
The eight notch triangles it fills in	0.2427 s^2	10.4%
The eight semicircular lobes	0.9201 s^2	39.4%
Composite floor plate	**2.3344 s^2**	**100%**

Pelli's addition **nearly doubled the floor plate** — a 99% gain — while leaving all eight points, and so the symbol, plainly visible from the air.

DOES THE MODEL MATCH THE REAL BUILDING? — THE BEST MOMENT IN THE LESSON

Each tower has roughly 218,000 m^2 of floor space over 88 floors, about **2,500 m^2 per floor**. The model says the plate is $2.3344s^2$, so $s \approx 33$ m.

Now check a number the model did *not* use. With $s = 33$ m the star points sit 23.3 m from the center, making the tower about **46.7 m across**. Published plans put the Petronas floor plate at roughly 46 m wide.

Two independent figures, one model, and they agree. Worth saying out loud to students: this is the difference between a model that describes a building and one that merely resembles it.

DESIGN IT YOURSELF

The lobe radius was a choice, not a law. Keep the star and let each lobe have radius r: the eight semicircles contribute $8 \times \frac{1}{2}\pi r^2 = 4\pi r^2$.

Have students try $r = 0.15s$, $0.20s$, and $0.2706s$ and compute the plate each time, then argue: bigger lobes mean more rentable space, but at some point the points vanish and the building stops reading as a Rub el Hizb. Where would you stop? There is no right answer, which is the point.

05 Practice, with answers

1. Two squares of side 20 m, one rotated 45°. Find the overlap octagon's area, then the star's.

ANSWER Overlap = $2(\sqrt{2} - 1)(400) \approx$ **331.4 m^2**. Star = 800 − 331.4 = **468.6 m^2**. Check: $(4 - 2\sqrt{2})(400) = 468.6$ m^2. ✓

2. Verify the star is exactly $\sqrt{2}$ times the octagon.

ANSWER 468.6/331.4 = **1.4142**. True for every s, since both are constant multiples of s^2 and the s^2 cancels.

3. For those squares, how far apart are neighbouring star points, and what is each lobe's radius?

ANSWER $R = 20/\sqrt{2} \approx 14.14$ m; chord = 2(14.14)(0.3827) ≈ **10.82 m**; lobe radius ≈ **5.41 m**.

4. Find the total lobe area for those squares, and the finished plate.

ANSWER Eight semicircles = $4\pi(5.41)^2 \approx$ **368 m^2**. With the star and the notch triangles (≈ 97 m^2): 468.6 + 97 + 368 ≈ **934 m^2** = 2.3344 × 400. ✓

5. A designer shrinks the lobes to radius $0.15s$. What happens to the plate?

ANSWER Semicircles = $4\pi(0.15s)^2 \approx 0.2827s^2$, so the plate falls to about **$1.697s^2$** from $2.334s^2$ — a **27%** loss. Halving the radius costs far more than half the lobe area, because area goes with the *square* of the radius.

6. Why is the star only 17% bigger than one square when it looks so much larger?

ANSWER The squares overlap heavily: the octagon is **41%** of their combined area. The eight points look dramatic but are thin — all eight together come to just $0.3431s^2$, about a third of one square. Eye-catching is not the same as space-efficient, which is the whole reason §4 exists.

BACK TO THE OPENING QUESTION

What fraction is star and what fraction is bulges? The star is just over **half** — 50.2% — with the lobes and the notches they fill making up the other **49.8%**. Almost exactly half the office space in the Petronas Towers sits in the rounded bays between the points.

That is the trade at the heart of the building. Mahathir's sketch supplied a symbol; the star alone would have made a beautiful, uneconomic tower. Eight semicircles doubled the rentable floor without erasing a single point of the Rub el Hizb — and the whole compromise is measurable with $A + B -$ overlap and the area of a circle.

Teacher's Guide · Google Earth Voyager Story: The Geometry of Polygon-Shaped Buildings, Part 2
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.