The Geometry of Polygon-Shaped Buildings

Interior angles, hexagon construction, and why only three regular polygons tile a plane

The Pentagon is five-sided. Now try to tile a floor with regular pentagons — you can't. There is always a gap. The size of that gap is a specific number, and Islamic artists in Marrakesh spent centuries turning it into something beautiful.

GRADE BAND	TIME	FORMAT
7–10 · Middle School through HS Geometry	Three class periods	Video, compass-and-straightedge construction, dynamic geometry, tiling investigation

WHAT STUDENTS WILL BE ABLE TO DO

- Find the interior, exterior, and central angles of any regular polygon from a single formula.
- Construct a regular hexagon with compass and straightedge — and explain why the construction works.
- Prove that only three regular polygons can tile a plane by themselves, and name them.
- Calculate the area of a regular polygon from its side length using the apothem.
- Explain how Islamic artists built patterns from shapes that can't tile on their own.

STANDARDS

7.G.A.2 drawing geometric shapes with given conditions · **8.G.A.5** angle relationships · **HSG-CO.A.3** symmetries of regular polygons · **HSG-CO.D.12** formal geometric constructions · **HSG-CO.D.13** constructing an equilateral triangle, square, and regular hexagon inscribed in a circle · **HSG-MG.A.1** and **A.3** modeling with geometry and design problems

START HERE

Open the Voyager story in Chrome, look down on the Pentagon, then fly to Marrakesh and zoom in on the tilework of a madrasa courtyard. Two five-sided things, an ocean and seven centuries apart.

Then hand out paper: draw a regular pentagon and try to surround it with identical pentagons, corner to corner, no gaps. *What goes wrong?* Have students measure the gap if they can. §6 names it exactly, and the hands-on failure first is what makes the proof land.

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01 Polygon-shaped buildings

VIDEO **Polygons: Introduction — the Pentagon**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PolygonsIntro.mp4

Polygon	A closed figure made of straight line segments.
Regular polygon	All sides congruent *and* all angles congruent. Both conditions are needed — worth stressing, since students often assume one implies the other.
Interior angle	The angle inside the polygon at a vertex.
Exterior angle	The angle you turn through at a vertex when walking the perimeter.
Central angle	The angle at the center between two adjacent vertices.
Apothem	Distance from the center to the middle of a side — the polygon's "inradius."

$$\text{interior} = (n-2)\cdot 180°/n \quad\cdot\quad \text{exterior} = \text{central} = 360°/n$$

REGULAR POLYGON	SIDES	INTERIOR ANGLE	CENTRAL ANGLE
Equilateral triangle	3	60°	120°
Square	4	90°	90°
Pentagon	5	108°	72°
Hexagon	6	120°	60°
Octagon	8	?	?
Decagon	10	?	?

Blank rows: octagon 135° and 45°; decagon 144° and 36°. Flag the decagon's 36° — it returns in §6 as the pentagon's tiling gap.

The Pentagon, by the numbers

Designed by George Bergstrom and built in sixteen months between 1941 and 1943, the Pentagon has five outer walls of **921 feet** each. The shape was not symbolic: the original site was bounded by five roads and the architects fitted the building to the land.

$$a = s/(2\tan 36°) = 921/1.453 \approx 634\ \text{ft}$$

$$\text{Area} = \tfrac{1}{2} \cdot (5 \times 921) \cdot 634 \approx 1{,}459{,}000\ \text{ft}^2 \approx \mathbf{33.5\ acres}$$

The published gross footprint is 33.5 acres — the formula lands on it exactly, a satisfying check that the building really is a regular pentagon and not merely pentagon-ish. Worth doing on the board.

> WHY FIVE SIDES MAKES IT FAST TO CROSS
>
> Inside, the Pentagon is **five concentric pentagonal rings** (A–E) joined by **ten radial spokes** — 17.5 miles of hallway, yet the DoD claims any two points are about seven minutes apart.
>
> Check it: the five-acre courtyard is a pentagon of side ≈ 356 ft, apothem 245 ft, so the building depth is 634 − 245 ≈ 389 ft. The longest sensible walk is in along a spoke, part-way around the inner ring, and out: 389 + 889 + 389 ≈ **1,667 ft**. At 3 mph that is **6.3 minutes**.
>
> The claim holds, and it holds because of the ring-and-spoke layout — the same radial geometry compared against city grids in the City Planning stories. If your class did those, this is the payoff.

02 Polygons and arabesques

VIDEO **Marrakesh, Segment 1 — the cultural legacy**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/MarrakeshSegment--GoogleVoyagerFull--Segment1.mp4

The tilework across Marrakesh — the Ben Youssef Madrasa, the Koutoubia minaret — is **zellij**: hand-cut glazed terracotta assembled into geometric mosaics. A craftsman chips each piece to shape from a fired tile, then sets them face-down into plaster.

That method has a mathematical consequence worth naming for students: every tile must be one of a small number of standard shapes cut to a template, or the workshop could never produce enough. The patterns look infinitely varied; underneath they are a handful of repeated polygons. (Same argument as the Colosseum's stone templates, if the class has done Circular Structures.)

03 Euclid's influence on Islamic artists

VIDEO **Marrakesh, Segment 2 — Greek ideas and Islamic scholars**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/MarrakeshSegment--GoogleVoyagerFull--Segment2.mp4

Euclid wrote the *Elements* in Alexandria around 300 BCE. In the ninth century it was translated into Arabic in Baghdad — first by al-Hajjaj ibn Yusuf, later in a more careful version associated with Thabit ibn Qurra. For several centuries the best mathematics in the world was being done in Arabic, and Euclid's constructions were standard education.

A BOOK WRITTEN FOR CRAFTSMEN

Around 990 CE the Persian mathematician **Abū al-Wafā' al-Būzjānī** wrote a work titled roughly *On Those Parts of Geometry Needed by Craftsmen* — a geometry book aimed at the people cutting the tiles.

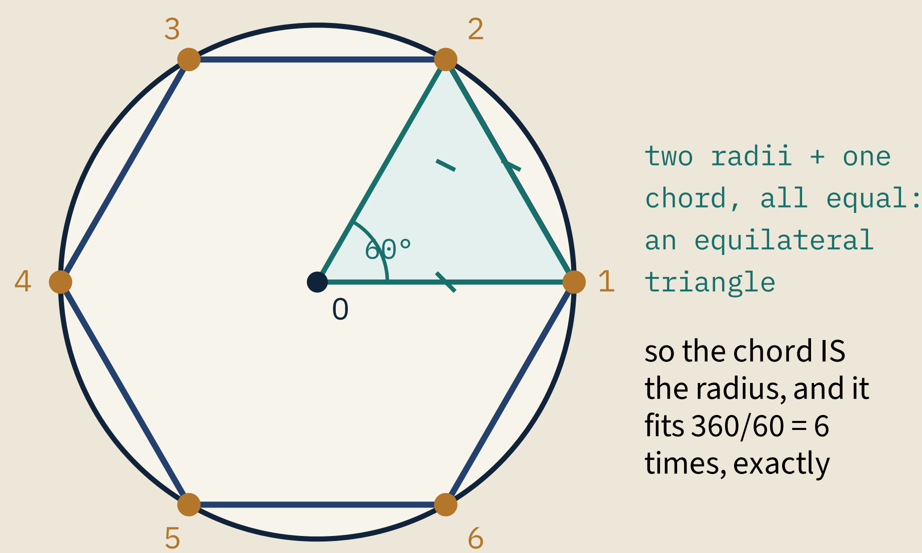
He records attending meetings between mathematicians and artisans where the two groups argued: craftsmen wanted constructions that worked on a workbench with a fixed compass, mathematicians wanted constructions that were provably exact. The book is his attempt to give artisans methods that were both.

Use this to head off the idea that the patterns are decoration that happens to look mathematical. They descend from a documented conversation between geometers and tile-cutters.

04 Constructing a regular hexagon

VIDEO **Marrakesh, Segment 3 — compass, ruler, pencil, paper**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/MarrakeshSegment--GoogleVoyagerFull--Segment3.mp4

Draw a circle, then without changing the compass setting, step that radius around the circumference. It lands exactly six times.



The construction isn't a lucky coincidence — it's an equilateral triangle six times over.

WHY IT WORKS — DON'T SKIP THIS

A hexagon's central angle is 360°/6 = 60°. The triangle formed by the center and two adjacent vertices has two radii as sides, so it is isosceles with a 60° apex — which forces base angles of 60° too. It is **equilateral**.

So the chord equals the radius. Stepping the compass radius around the circle marks off exactly 60° each time, and 360/60 = 6 lands precisely back at the start. No other regular polygon has this property, which is why the hexagon is the one students can construct in ten seconds.

05 The same construction with technology

VIDEO **Marrakesh, Segment 4 — the construction on the TI-Nspire**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/MarrakeshSegment--GoogleVoyagerFull--Segment4.mp4
Any dynamic geometry software works.

DRAG THE CENTER

Once the hexagon is built, have students drag the original circle bigger and smaller. Every vertex moves; every side stays equal to the radius; every angle stays 120°.

That is the difference between *drawing* a hexagon and *constructing* one. A drawing is a picture that happens to look right; a construction is a set of relationships that survives being pulled on — which is exactly what Abū al-Wafā's craftsmen needed from a template.

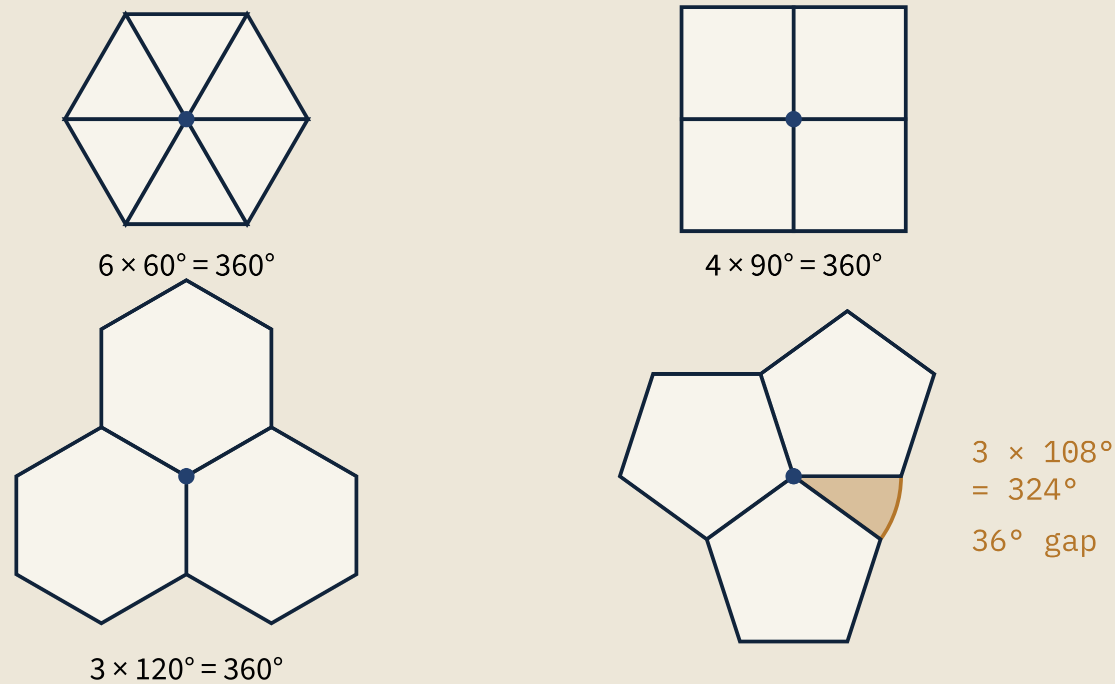
06 Exploring tessellations

VIDEO **Marrakesh, Segment 5 — tessellations**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/MarrakeshSegment--GoogleVoyagerFull--Segment5.mp4

A **tessellation** covers a plane with no gaps and no overlaps. The deciding question: which regular polygons can do it alone? Look at any point where tiles meet — the angles there must total exactly 360°.

360°/interior angle must be a whole number

REGULAR POLYGON	INTERIOR ANGLE	360 ÷ INTERIOR	TILES ALONE?
Triangle	60°	6	Yes — six meet at a point
Square	90°	4	Yes — four meet at a point
Pentagon	108°	3.33…	No
Hexagon	120°	3	Yes — three meet at a point
Heptagon	128.57°	2.8	No
Octagon	135°	2.67	No
Decagon	144°	2.5	No



Three pentagons fall 36° short. That gap is the whole story.

> ONLY THREE, AND WHY THERE CAN'T BE A FOURTH
>
> For $n \geq 7$ the interior angle exceeds 120° but stays under 180°. So *two* leave a gap and *three* overlap — no room for a third, no way to close with two. Below that, only $n = 3$, 4, and 6 divide 360 evenly. The pentagon is the single case in between that just misses, by exactly **36°**.
>
> This is a complete proof at middle-school level, and students rarely get one. Let them state it themselves after the table.

What the artists did with the gap

Islamic geometric patterns are full of pentagons, decagons, and ten-pointed stars — shapes that provably cannot tile alone. The artists were not defeated by that; they used it. If a pentagon leaves 36°, fill it with a different tile. Patterns built this way combine two, three, or four template shapes and repeat forever without a gap, while never being as monotonous as a grid of squares.

Note the number: 36° is one tenth of a full turn and the central angle of a regular decagon. The gap is not random left-over space — it is itself clean polygon geometry, which is exactly what makes it usable.

07 Practice, with answers

1. Fill in the blank rows: interior and central angles of a regular octagon and decagon.

> ANSWER Octagon: **135°** and **45°**. Decagon: **144°** and **36°**. The decagon's central angle is the pentagon's tiling gap.

2. A regular hexagonal gazebo has 8 ft sides. Find its apothem and floor area.

> ANSWER Apothem = 8/(2 tan 30°) ≈ **6.93 ft**. Area = ½ × 48 × 6.93 ≈ **166 ft²**. Check with the hexagon shortcut $2.598s^2$: 2.598 × 64 ≈ 166 ft². ✓

3. A regular polygon's interior angle is 156°. How many sides?

> ANSWER Exterior = 180 − 156 = 24°, and exterior = 360/*n*, so *n* = **15**. Going through the exterior angle is almost always faster than solving the interior formula.

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4. Could a regular 12-gon tile a plane by itself?

> ANSWER Interior = 150°; 360/150 = 2.4, so **no**. Two dodecagons leave a 60° gap — exactly an equilateral triangle. That combination is a real tiling, just not from a single shape. A good bridge to semi-regular tilings.

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5. The Pentagon's outer wall is 921 ft. How far is it around the outside, and how does that compare with 17.5 miles of corridor?

> ANSWER Perimeter = **4,605 ft** ≈ 0.87 miles. The corridors run about **twenty times** the building's own perimeter — what stacking five rings and five floors will do.

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6. Why can't a regular 100-gon tile the plane?

> ANSWER Interior = 176.4°. Two come to 352.8° (a 7.2° gap); three come to 529°, well past a full turn. Any polygon with seven or more sides is trapped the same way: too big for three, too small for two.

> BACK TO THE OPENING QUESTION
>
> What went wrong surrounding a pentagon with pentagons? Three fit around a point and total 324°, leaving a wedge of exactly **36°**. Only the triangle, square, and hexagon divide 360° evenly, so only those three tile a plane alone.
>
> The Pentagon in Arlington never had to worry about it — one building, no tiling required. The tile-cutters of Marrakesh had no such luxury, and their answer was better than avoiding pentagons: they kept the pentagons and made the 36° gap part of the design.

Teacher's Guide · Google Earth Voyager Story: The Geometry of Polygon-Shaped Buildings, Part 1
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.