TEACHER'S GUIDE · GOOGLE EARTH VOYAGER STORY · PART 2 OF 2

The Mathematics of Pyramids

Mayan stepped pyramids: sequences, series, and exponential functions

El Castillo at Chichén Itzá has 365 steps. One for every day of the year. That is not a coincidence, and it is not the only number the Maya built into the stone — but finding the rest takes sequences, series, and a sum.

GRADE BAND	TIME	FORMAT
9–12 · Algebra 2 and Precalculus	**Two class periods**	**Video, calculator modeling, data investigation**

Part 1 covered smooth square pyramids — one solid, four triangular faces, a single apex. Mayan pyramids are built differently. They are **stair-step pyramids**: a stack of square or rectangular slabs, each smaller than the one beneath it.

That difference is the whole lesson, and it is worth saying to students explicitly. A smooth pyramid is one shape measured with one formula. A stepped pyramid is a *list* of shapes, and measuring it means adding the list up. Lists that shrink by a constant factor are geometric sequences; adding them is a geometric series.

WHAT STUDENTS WILL BE ABLE TO DO

- Tell a geometric sequence from an arithmetic one, and say which one a given feature of a pyramid follows.
- Write the explicit formula for a geometric sequence and evaluate any term.
- Read and write summation notation, and evaluate a finite geometric series by hand and on a calculator.
- Build a model of a real stepped pyramid from a handful of measurements, then test it against the building.
- Explain why the Maya place-value system is a geometric sequence — and find the one place where it deliberately isn't.

STANDARDS

HSF-BF.A.2 writing arithmetic and geometric sequences recursively and explicitly · **HSA-SSE.B.4** deriving and using the sum of a finite geometric series · **HSF-LE.A.1** distinguishing linear from exponential growth · **HSF-LE.A.2** constructing exponential functions · **HSF-IF.C.7e** graphing exponential functions · **HSG-MG.A.1** modeling real objects with geometric shapes

START HERE

Open the Voyager story and fly to Palenque and Chichén Itzá (best viewed in Chrome). Have students look straight down, then from the side, and count everything countable: sides, staircases, steps, terraces.

Collect the numbers on the board. Then ask: *which of these could be an accident, and which one couldn't?* Leave the list up — §5 settles it.

earth.google.com/web/data=Mj8KPQo7CiExbFNTNlFMZVFrMk8tM0N4NVVmbFFkN0I2SksyenA3WDkSFgoUMEY1MDVFN0NBODEyRDczNTU0RDg

REVIEW **Brief Review: Sequences and Series**
media4math.com/Brief-Review-Sequences-Series
Assign ahead if students haven't met sigma notation.

01 The Temple of the Inscriptions

VIDEO **Palenque, Segment 1 — the geometry of the Temple of the Inscriptions**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PalenqueSegmentGoogleVoyager--Segment1.mp4

The Temple of the Inscriptions was built at Palenque in the 7th century as a tomb for K'inich Janaab' Pakal, who ruled for 68 years. Numbers are worked into it everywhere: eight stepped tiers with the temple on top makes nine levels, matching the nine levels of the Maya underworld; 69 steps climb the front; three carved panels inside hold 617 hieroglyphs, one of the longest Maya texts known.

HOW TALL IS IT? — A MODELING DISCUSSION WORTH HAVING

Published heights for this pyramid range from 20 m to 27 m. The sources aren't sloppy; they're measuring different things. Some stop at the temple platform, some include the temple, and the back of the pyramid is built into a hillside so "ground level" depends on which side you stand on.

Use this before the modeling work in §4, not after. Students who have argued about what counts as the height are much less likely to treat the numbers in §4 as facts handed down rather than choices made.

Summation notation

Adding up the tiers of a stepped pyramid needs a compact way to write "add these terms" — the Greek capital sigma:

$$\sum_{n=1}^{9} a_n$$

Read: "the sum of a sub n, from n equals 1 to 9." Three parts do the work — the **index** under the sigma, the **start and stop** values, and the **rule** to the right that generates each term.

Sequence — An ordered list of numbers. Each one is a *term*.

Series — The sum of the terms of a sequence.

Arithmetic sequence — Each term is the previous term *plus* a constant, d. Straight-line growth.

Geometric sequence — Each term is the previous term *times* a constant, r. Exponential growth or decay.

Common ratio — The multiplier. Find it by dividing any term by the one before it.

Index — The counter under the sigma that steps through the terms.

$$a_n = a_1 \cdot r^{n-1}$$

02 Geometric sequences and pyramids

VIDEO **Palenque, Segment 2 — summing a sequence on the TI-Nspire**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PalenqueSegmentGoogleVoyager--Segment2.mp4

Nine terms can be added by hand. Ninety cannot, and that is the argument for the formula. For a finite geometric series:

$$S_n = a_1 \cdot (1 - r^n)/(1 - r)$$

Every symbol is something students can read off a building: a_1 is the bottom tier, r is how much each tier shrinks, n is how many tiers there are.

```
TI-Nspire
  Generate terms:   seq(55.3*0.88^(n-1), n, 1, 9)
  Sum them:         sum(seq(55.3*0.88^(n-1), n, 1, 9))
  Sigma template:   Σ(55.3*0.88^(n-1), n, 1, 9)    [ menu > Calculus > Sum ]
  Graph the model:  f1(x) = 55.3*0.88^x

Desmos equivalent
  Type "sum" to insert the sigma template; enter the
  function on its own line to graph it.
```

ASK BEFORE COMPUTING

The graph of $f(x) = 55.3 \cdot 0.88^x$ is a smooth curve, but a pyramid has nine tiers and no halves. Which points on the curve are real terraces, and which are the curve doing what curves do?

This is the discrete-versus-continuous distinction, and a graphing calculator makes it easy to blur. Have students mark the nine points.

03 The Mayan pyramid of Chichén Itzá

VIDEO **Chichén Itzá, Segment 3 — exploring El Castillo**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PalenqueSegmentGoogleVoyager--Segment3.mp4

El Castillo — the Temple of Kukulcán — is a nine-terrace stepped pyramid with a square base 55.3 m on a side, rising 24 m to the temple platform and 30 m counting the temple. Four staircases, one per cardinal direction, climb 91 steps each.

TWO DIFFERENT SEQUENCES IN ONE BUILDING

The **steps** are arithmetic: each one lifts you the same distance as the last. Constant difference, straight-line growth.

The **terraces** are geometric: each is a constant *fraction* of the width below it. Constant ratio, exponential decay.

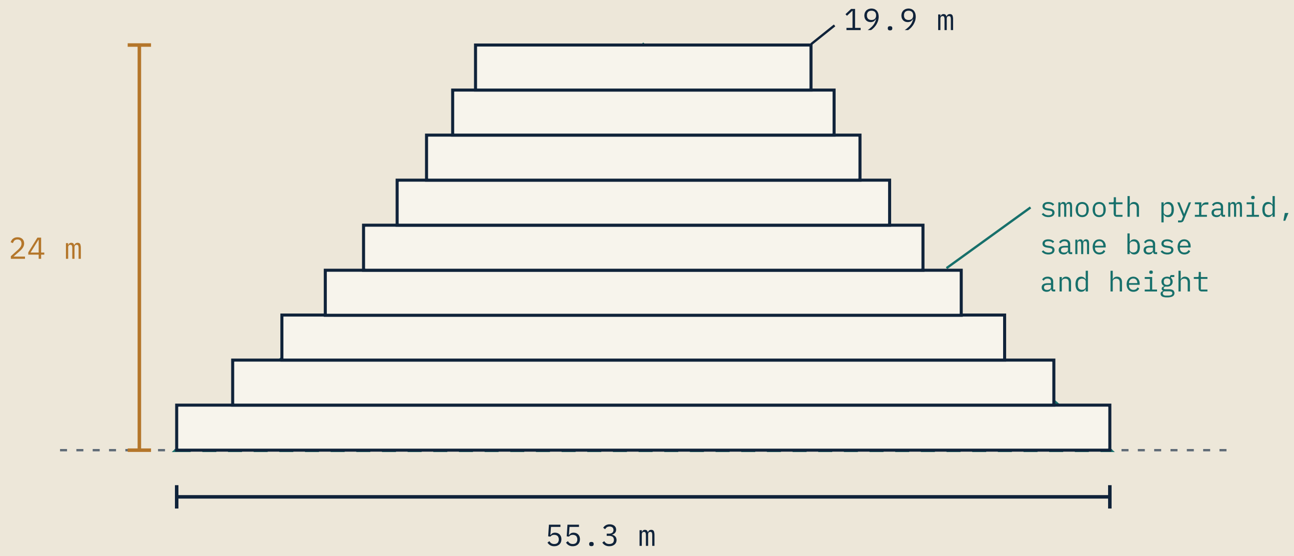
Same building, two kinds of pattern. The diagnostic students should internalize: do consecutive terms differ by a constant *amount* or a constant *factor*?

04 Modeling Chichén Itzá

VIDEO **Chichén Itzá, Segment 4 — building the model on the TI-Nspire**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PalenqueSegmentGoogleVoyager--Segment4.mp4

The model: treat each of the nine terraces as a square slab of equal thickness, with side lengths forming a geometric sequence. The bottom terrace is 55.3 m across, the top platform is close to 19.9 m, and 24 m of height split nine ways gives each slab a thickness of about 2.67 m.

Nine terms, first to last, means eight multiplications by r. So $r^8 = 19.9/55.3 \approx 0.36$, giving $r \approx 0.88$ — each terrace about 88% as wide as the one under it.



Nine terraces, each 0.88 as wide as the one below. Drawn to scale.

TERRACE N	SIDE S_N (M)	AREA S_N^2 (M²)	RUNNING TOTAL (M²)
1	55.30	3,058.1	3,058.1
2	48.66	2,368.2	5,426.3
3	42.82	1,833.9	7,260.2
4	37.69	1,420.2	8,680.4
5	?	?	?
6	?	?	?
7	25.68	659.5	11,291.3
8	22.60	510.7	11,802.0
9	19.89	395.5	12,197.5
Total surface stacked			**12,198 m²**

Blank rows: terrace 5 is 33.16 m / 1,099.8 m² / 9,780.2 m²; terrace 6 is 29.18 m / 851.6 m² / 10,631.8 m².

Note what happened to the ratio. The side lengths shrink by 0.88, but the *areas* shrink by $0.88^2 = 0.7744$. Squaring a geometric sequence gives another geometric sequence with the ratio squared — so the areas are their own geometric series and the formula applies directly:

$$3{,}058.1 \cdot (1 - 0.7744^9)/(1 - 0.7744) \approx 12{,}198 \text{ m}^2$$

$$V \approx 12{,}198 \times 2.67 \approx 32{,}600 \text{ m}^3$$

COMPARE IT TO PART 1

A *smooth* pyramid on the same 55.3 m base and 24 m height holds $\frac{1}{3} \times 3{,}058 \times 24 \approx 24{,}500$ m^3. The stepped pyramid holds about 32,600 m^3 — roughly a third more stone on exactly the same footprint.

That is the dashed triangle in the diagram. Every place a terrace juts past the dashed line is material the smooth pyramid doesn't have. If students did Part 1, this is where the ⅓ from that lesson earns its keep.

05 Mayan pyramids and Mayan math

VIDEO **Chichén Itzá, Segment 5 — Mayan math and the Mayan universe**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PalenqueSegmentGoogleVoyager--Segment5.mp4

The Maya wrote numbers in base 20 using three symbols: a shell for zero, a dot for one, a bar for five. Theirs is one of the earliest number systems anywhere to treat zero as a genuine placeholder.

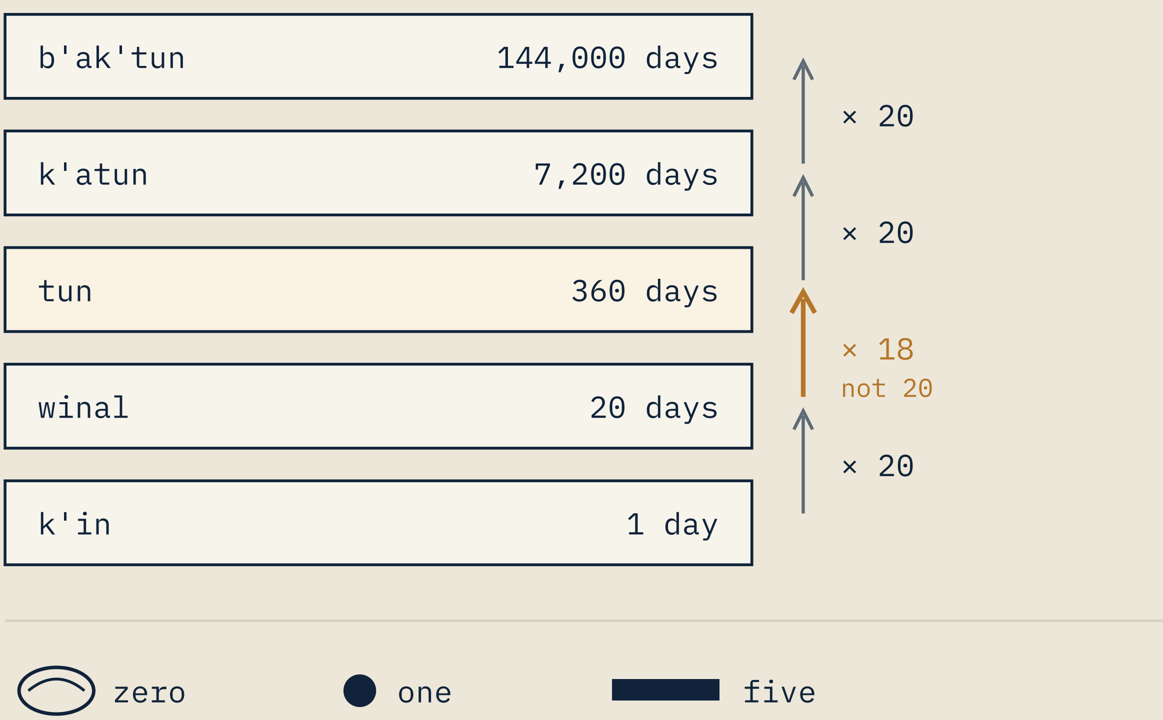
Base 20 means the place values are powers of 20 — and powers of a constant are exactly what a geometric sequence is. Our own place values are the same idea with $r = 10$:

$$1, 20, 400, 8{,}000, 160{,}000, \ldots = 20^0, 20^1, 20^2, 20^3, 20^4$$

Which makes any place-value system an exponential function in disguise: $f(x) = 20^x$, sampled at the whole numbers.

Where the pattern breaks on purpose

The Maya used pure base 20 for counting things. For counting *days* — the Long Count calendar — they changed one rung of the ladder.



The Long Count ladder. Every rung multiplies by 20 except one.

A *tun* is 18 winals, not 20 — 360 days instead of 400. The reason is written in the sky: 360 is far closer to a solar year than 400. The Maya deliberately broke a clean mathematical pattern to keep the calendar tied to the sun, then resumed multiplying by 20 for every rung above.

ARGUE IT OUT

Was breaking the pattern a good decision? Have students build the case both ways. What does the Long Count gain from a third place of 360? What does it cost — consider converting a Long Count date to a plain day count, or teaching the system to a child who has just learned to count by twenties.

There is no settled answer here, which is the point: it is a design trade-off, and students who can name both sides of one are ready for the modeling questions that fill the rest of this course.

06 Practice, with answers

1. Write the explicit formula for the side length of terrace *n* at El Castillo, then find terrace 5.

ANSWER $s_n = 55.3 \cdot (0.88)^{n-1}$. For $n = 5$: $55.3 \times 0.88^4 = 55.3 \times 0.5997 \approx$ **33.2 m**. Terrace 6 is $55.3 \times 0.88^5 \approx 29.2$ m.

2. Write the total stacked area of all nine terraces in sigma notation.

ANSWER $\sum$ from n=1 to 9 of $(55.3 \cdot 0.88^{n-1})^2$, or equivalently $3{,}058.1 \cdot 0.7744^{n-1}$. Both give about 12,198 m². The second form makes the common ratio of the *areas* visible.

3. A staircase has 91 steps and climbs 24 m. What kind of sequence are the step heights, and what is the constant?

ANSWER **Arithmetic.** Each step adds the same height: $24 \div 91 \approx 0.264$ m, about 26 cm per step. This is a common difference, not a common ratio — with a geometric ratio below 1 the staircase would flatten out before reaching the top.

4. In a pure base-20 system, what is the fifth place value? In the Long Count, what is the fifth unit? Why don't they match?

ANSWER Pure base 20: 20^4 = **160,000**. Long Count: the b'ak'tun = **144,000** days. The gap comes entirely from the tun being $18 \times 20 = 360$ rather than 400. Check: $144{,}000/160{,}000 = 0.9 = 18/20$.

5. Suppose a stepped pyramid has terraces with ratio $r = 0.95$ instead of 0.88, same base and same nine terraces. Is it closer to or further from a smooth pyramid?

ANSWER **Further.** With $r = 0.95$ the terraces barely shrink, so the solid is closer to a rectangular box than a pyramid. The top terrace would be $55.3 \times 0.95^8 \approx 36.7$ m across, nearly two-thirds of the base. The smaller the ratio, the faster the sides pull in.

6. Extension: an infinite tower has terraces of side 55.3, 48.66, 42.82, … forever. Does its total stacked area converge?

ANSWER **Yes.** Because the area ratio 0.7744 is less than 1, the infinite series converges to $a_1/(1 - r) = 3{,}058.1/0.2256 \approx 13{,}556$ m². Infinitely many terraces, finite total — and the nine real ones already account for 90% of it.

BACK TO THE OPENING QUESTION

Which numbers were accidents? Almost none. Each of El Castillo's four staircases has 91 steps: 91 × 4 = 364, plus the top platform makes **365**, the days in the Haab' year. The nine terraces are split by each staircase into **18** sections — the 18 months of that calendar. Each face carries **52** recessed panels, the years in a full Calendar Round.

The Maya built a calendar you can climb. Every number the class counted from orbit was somebody's arithmetic, done a thousand years ago and set in limestone.

Teacher's Guide · Google Earth Voyager Story: The Mathematics of Pyramids, Part 2
Companion to the online lesson. Videos, animations, and the Voyager story are linked by URL above; the interactive version carries them inline.