The Mathematics of Pyramids

Nets, Platonic solids, and why a pyramid is one third of a box

A pyramid is one of the first solids students ever learn to name. It is also a shape Egyptian builders had to understand well enough to raise 2.3 million blocks of stone into a figure whose four faces meet at a single point, 146 meters up.

GRADE BAND	TIME	FORMAT
6–8, extensions for HS Geometry	**Two class periods**	**Video, hands-on build, data investigation**

WHAT STUDENTS WILL BE ABLE TO DO

- Name the faces, edges, and vertices of a cube and a square pyramid, and explain why the cube is a Platonic solid and the pyramid is not.
- Fold a net into a solid, and predict which nets work before cutting.
- Explain *why* the volume of a pyramid is one third the volume of a prism with the same base and height — not just recite the formula.
- Use the Pythagorean Theorem to find the slant height of a real pyramid, then calculate volume and surface area.

STANDARDS

6.G.A.4 nets and surface area · **7.G.B.6** area, volume, and surface area of three-dimensional objects · **8.G.B.7** Pythagorean Theorem in three dimensions · **HSG-GMD.A.1** informal arguments for volume formulas · **HSG-GMD.A.3** using volume formulas · **HSG-MG.A.1** modeling real objects with geometric shapes

START HERE

Fly to Giza first. Open the Google Earth Voyager story and watch the slide show. As students look down on the plateau, pose one question: *how would you have checked that the square base was actually square?* Park the answers on the board — the lesson returns to them at the end.

earth.google.com/web/data=MikKJwolCiExbFNTNlFMZVFrMk8tM0N4NVVmbFFkN0I2SksyenA3WDkgAUICCABKCAj5q6vyBxAB

01 The family a pyramid belongs to

Pyramids belong to a large group of three-dimensional figures called **polyhedra** — solids whose faces are all flat polygons. The opening video introduces the most exclusive club inside that group: the Platonic solids.

VIDEO **3D Geometry: Introduction**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/3DGeo--Intro.mp4

Polyhedron	A solid whose surface is made entirely of flat polygons.
Face, edge, vertex	A flat surface, a line where two faces meet, and a point where edges meet.
Platonic solid	A polyhedron whose faces are all congruent regular polygons, with the same number of faces meeting at every vertex. There are exactly five.
Net	A flat pattern that folds into a solid without gaps or overlaps.

Apex	The single point where all the triangular faces of a pyramid meet.

Only five Platonic solids exist — a fact proved by the ancient Greeks and still true today. Counting the parts of each one turns up something remarkable in the last column.

SOLID	FACES (F)	VERTICES (V)	EDGES (E)	V − E + F
Tetrahedron	4	4	6	2
Cube (hexahedron)	6	8	12	2
Octahedron	8	6	12	2
Dodecahedron	12	20	30	2
Icosahedron	20	12	30	2
Square pyramid	?	?	?	?

That pattern is **Euler's formula**, and it holds for every convex polyhedron, Platonic or not:

$$V - E + F = 2$$

Have students fill in the last row for a square pyramid and check whether Euler's formula still works. *(5 faces, 5 vertices, 8 edges → 5 − 8 + 5 = 2. It does.)*

02 Building cubes and pyramids

A cube has six square faces, and every side length and angle measure is congruent. As a Platonic solid it is called a **hexahedron**, which literally means "six sided."

A square pyramid is *not* a Platonic solid, even though it looks like it belongs. It can share a base with a cube and stand the same height, but its faces are not all congruent — four triangles and one square — and its apex has four faces meeting there while each base vertex has only three.

	CUBE	SQUARE PYRAMID
Faces	6 squares	1 square + 4 triangles
Edges	12	8
Vertices	8	5
Faces at each vertex	3 everywhere	3 at the base, 4 at the apex
Platonic?	Yes	No

VIDEO **3D Geometry Animation: Cube · and · Square Pyramid**
s3.amazonaws.com/Media4MathPlus/Videos/GeometryVideos/3DGeometryCube.mp4
s3.amazonaws.com/Media4MathPlus/Videos/GeometryVideos/3DGeometryPyramid.mp4
Rotating views: top, side, and bottom of each figure.

Fold your own

Print the nets, or have students draw their own on grid paper. The printable sheet includes glue flaps.

HANDOUT **Printable Nets (PDF)**
media4math.com/sites/default/files/library_asset/files/PrintableNets.pdf

ANIMATION **Net for Cube · and · Net for Square Pyramid**
media4math.com/library/animated-math-clip-art-3d-geometry-net-cube
media4math.com/library/animated-math-clip-art-3d-geometry-net-square-pyramid
Project these before students crease anything. On the pyramid, ask them to watch where the four triangles meet.

PREDICT BEFORE YOU CUT

There are eleven distinct nets that fold into a cube. Have students sketch three arrangements of six squares on grid paper, predict which will close up, then cut and test. What goes wrong in the ones that fail?

Then the pyramid: a square with four triangles attached. Does *every* arrangement of those five polygons fold into a pyramid?

03 Why one third?

Students can watch the video or build along with it. Materials for the build:

- Cubes for constructing the pyramids — all the same size and color, with no labels on them.
- Grid paper and a pencil.
- A recording sheet or a spreadsheet.

VIDEO **Exploring the Volume of a Pyramid**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PyramidVolume.mp4

The stacked-cube investigation

Build a stepped pyramid out of cubes. The top layer is 1 cube. The layer under it is a 2 × 2 square. Under that, 3 × 3. Each time a layer is finished, record two numbers: the total cubes used, and the number of cubes needed to fill a *cube-shaped box* with the same base and height.

LAYERS (N)	BOTTOM LAYER (N^2)	CUBES USED	BOX HOLDS (N^3)	CUBES ÷ BOX
1	1	1	1	1.000
2	4	5	8	0.625
3	9	14	27	0.519
4	16	30	64	0.469
5	25	55	125	0.440
6	?	?	?	?
10	?	?	?	?
100	10,000	338,350	1,000,000	0.338

The last column is falling toward something. It never quite arrives — a stepped pyramid always has extra stone hanging off each step — but the thinner the steps, the closer it gets to **0.333…**

THE RESULT

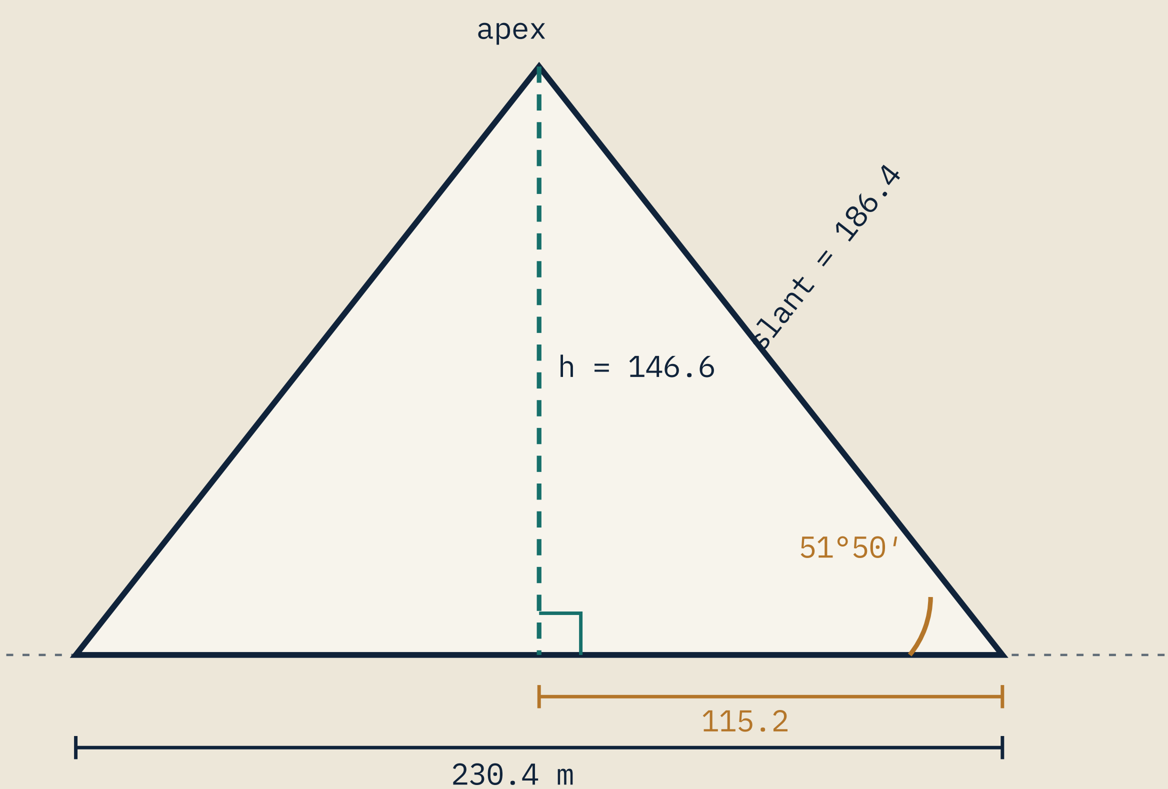
Three identical pyramids, each with the same base and height as a given prism, fill that prism exactly. So:

$$V = \tfrac{1}{3} Bh$$

where *B* is the area of the base and *h* is the height measured straight up from the base to the apex — not along a slanted face. That last distinction is the most common error in this unit; name it out loud.

04 The Great Pyramid, by the numbers

Now point the formula at something real. The Great Pyramid of Khufu was built around 2560 BCE. Erosion and the loss of its outer casing stones have shrunk it, so these are its *original* dimensions, in meters.



Cross-section through the middle of two opposite faces. Drawn to scale.

The height and the half-base form a right triangle, so the slant height — the distance from the middle of one base edge straight up the face to the apex — comes from the Pythagorean Theorem:

$$\sqrt{(146.6^2 + 115.2^2)} = \sqrt{34{,}762.6} \approx 186.4 \text{ m}$$

And the volume:

$$\tfrac{1}{3} \times 230.4^2 \times 146.6 \approx 2{,}594{,}000 \text{ m}^3$$

Roughly 2.3 million blocks went into that volume, which works out to a little over one cubic meter of stone per block — about 2.5 metric tons each.

> HOW THE EGYPTIANS SAID "51°50′"
>
> Egyptian builders had no protractors and no trigonometry. They described slope with a single number, the **seked**: how far you move horizontally for every one cubit you rise. One cubit is 7 palms, and the Great Pyramid's seked is 5½ palms.
>
> So its slope is 7 rise over 5.5 run — and 7 ÷ 5.5 ≈ 1.2727, exactly the ratio 146.6 ÷ 115.2 above. A whole-number recipe a work crew could follow, block by block, without ever naming an angle.

05 Practice, with answers

1. A square pyramid has a base 12 cm on a side and a height of 9 cm. Find its volume. How does it compare to a box with the same base and height?

> ANSWER $V = ⅓ \times 12^2 \times 9 = ⅓ \times 144 \times 9 =$ **432 cm³**. The box holds 144 × 9 = 1,296 cm^3, exactly three times as much.

2. Find the slant height of that same pyramid, then its total surface area.

> ANSWER Half the base is 6 cm, so the slant height is $\sqrt{(9^2 + 6^2)} = \sqrt{117} \approx 10.8$ cm. Each triangular face is ½ × 12 × 10.8 ≈ 64.9 cm^2, so four faces ≈ 259.6 cm^2. Adding the 144 cm^2 base gives **≈ 403.6 cm²**.

3. Complete the missing rows of the stacked-cube table for *n* = 6 and *n* = 10.

> ANSWER Six layers: 36 in the bottom layer, 91 cubes total, a box of 216, ratio 0.421. Ten layers: 100 in the bottom layer, 385 cubes total, a box of 1,000, ratio 0.385. The totals follow $n(n+1)(2n+1) \div 6$.

4. A second pyramid has a seked of 5¼ palms. Is it steeper or shallower than the Great Pyramid?

> ANSWER **Steeper.** A smaller seked means less horizontal run per cubit of rise: 7 ÷ 5.25 ≈ 1.333, an angle of about 53.1° compared with 51.8°.

5. Double every dimension of a square pyramid. What happens to its volume?

> ANSWER It becomes **eight times** larger. The base area quadruples and the height doubles, so $⅓Bh$ grows by a factor of 4 × 2 = 8.

> BACK TO THE OPENING QUESTION
>
> How do you check that a 230-meter square is actually square, using only rope? Egyptian surveyors were called *harpedonaptai* — "rope-stretchers." A loop knotted into 12 equal segments pulls taut into a 3-4-5 triangle, and a 3-4-5 triangle contains a perfect right angle. Try it with string.
>
> **Coming in Part 2:** surface area, the mass of the stone, and what the ratio of the Great Pyramid's height to its base has to do with π.

Teacher's Guide · Google Earth Voyager Story: The Mathematics of Pyramids, Part 1
Companion to the online lesson. Videos, animations, and the Voyager story are linked by URL above; the interactive version carries them inline.