The Geometry of City Planning

Slope criteria, Pascal's triangle on a street map, and why the middle corner is busiest

There are 126 different shortest routes from one corner of a downtown grid to another. So if a hundred people set out at once, do they spread evenly across all 126? Not remotely — and the pattern that says why is Pascal's triangle.

GRADE BAND	TIME	FORMAT
7–9, extensions for HS Geometry	**Two class periods**	**Video, dynamic geometry construction, counting investigation**

Part 1 covers the shortest-distance principle, taxicab distance, and the comparison with a circular grid. This part takes up parallel and perpendicular lines, and the counting question Part 1 opened.

WHAT STUDENTS WILL BE ABLE TO DO

- Use the slope criteria for parallel and perpendicular lines, and apply them to a grid that isn't aligned to north.
- Count the shortest routes across a grid with combinations — and get the same answer by addition.
- Recognize Pascal's triangle sitting on a street map.
- Work out which intersections carry the most foot traffic, and by how much.
- Identify the angle pairs formed when a diagonal avenue cuts a grid, and calculate what the diagonal saves.

STANDARDS

8.EE.B.6 slope · **8.G.A.5** angles formed by parallel lines cut by a transversal · **HSG-CO.C.9** theorems about lines and angles · **HSG-GPE.B.5** slope criteria for parallel and perpendicular lines · **HSS-CP.B.9(+)** permutations and combinations · **HSG-MG.A.3** solving design problems with geometry

START HERE

Open the Voyager story in Chrome and zoom to street level on a busy downtown grid. Have students look at sidewalk widths, traffic lights, and worn paving, then ask: *if every route across the grid is the same length, why isn't every corner the same?*

§2 answers it with a number, and the answer is the best thing in this lesson.

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01 Parallel and perpendicular lines

VIDEO **Points and Lines, Segment 4 — parallel and perpendicular lines**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PointsLines--GoogleVoyagerSegment4.mp4
TI-Nspire in the video; desmos.com/geometry works equally well.

Dragging lines on a screen shows what parallel and perpendicular *look* like. Slope tells you how to *test* for them without looking — and the lesson never states the criteria, so state them.

Slope	Rise over run: how far a line climbs per unit travelled sideways.
Parallel lines	**Equal slopes**, $m_1 = m_2$.
Perpendicular lines	Slopes are **negative reciprocals**: $m_1 \cdot m_2 = -1$.
Transversal	A line cutting across two or more other lines.

$$\text{parallel: } m_1 = m_2 \quad \cdot \quad \text{perpendicular: } m_1 m_2 = -1$$

A GRID THAT ISN'T ALIGNED TO NORTH STILL OBEYS THIS

Houston's downtown grid, from Part 1, is rotated to follow Buffalo Bayou. If one set of streets runs with slope ¾, the cross streets must have slope −4⁄3, because ¾ × (−4⁄3) = −1.

Every street in the city then has one of exactly two slopes, and those two numbers multiply to −1. That single equation holds the whole plan together whichever way the city faces — a useful antidote to the idea that "perpendicular" means "horizontal and vertical."

Note the numbers too: a 3-4-5 right triangle, the same one that turns up in the Flatiron's floor plan and in Part 1's route calculation.

02 Why rectangular grids are used

VIDEO **Points and Lines, Segment 5 — why rectangular grids**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PointsLines--GoogleVoyagerSegment5.mp4

The student lesson asks students to verify that three drawn routes are the same length, then never gives the answer: all three are **9 units**. Getting from A to B takes four steps north and five east however you shuffle them, so every route is 4 + 5 = 9.

How many routes are there?

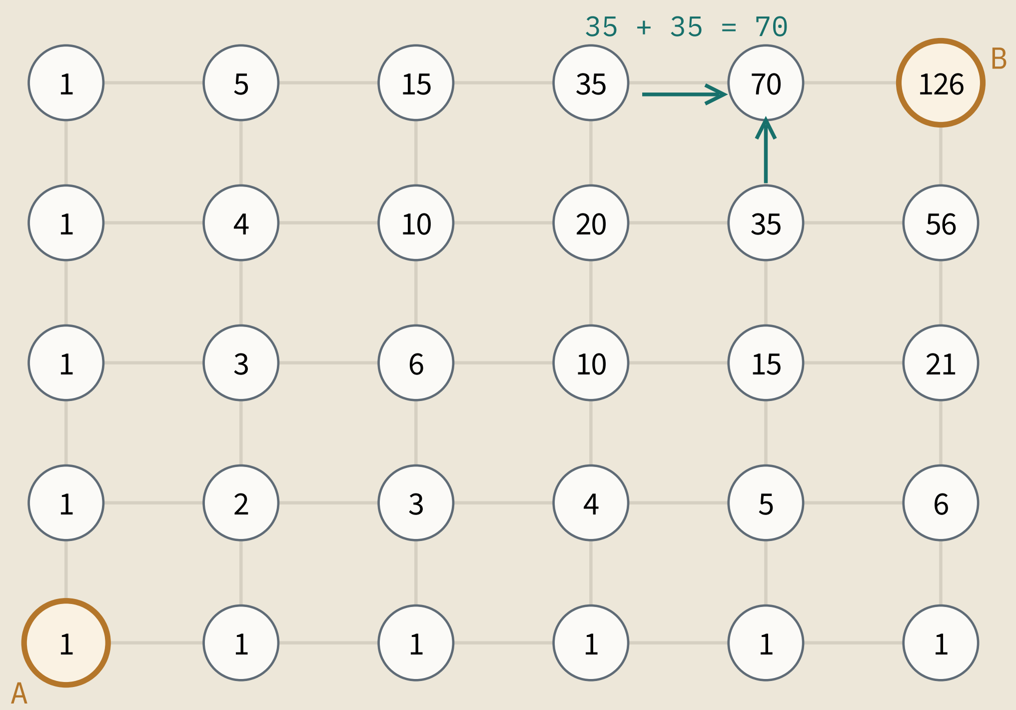
N N N N E E E E E

Choosing a route means choosing *which four* of the nine steps are north:

$$C(9, 4) = 9!/(4! \cdot 5!) = \mathbf{126}$$

The same answer, by addition

Students don't need the formula. Write 1 at A and fill in each intersection with the number of routes reaching it. Any corner can only be entered from the west or the south, so *routes to a corner* = *routes from the west* + *routes from the south*. Fill the bottom row and left column with 1s and add across the map.



Every intersection is the sum of the two before it. That's Pascal's triangle, rotated onto a street map.

The top-right corner reads **126** — the combination formula's answer, reached with nothing but addition. The numbers on that map *are* the binomial coefficients: a city grid is Pascal's triangle turned forty-five degrees.

But the routes are not equally used

The student lesson concludes that 126 routes means "just a handful of people are on any given path." That is worth pushing on, because it is the wrong picture.

Count the routes through an intersection by multiplying (routes from A to it) × (routes from it to B):

INTERSECTION	ROUTES THROUGH IT	SHARE OF TRAFFIC	OF 50 PEDESTRIANS
Middle of the grid	60	47.6%	≈ 24 people
One step off-center	45	35.7%	≈ 18 people
Halfway along an edge	21	16.7%	≈ 8 people
Far corner of the rectangle	1	0.8%	≈ 0.4 people

EQUAL LENGTH IS NOT EQUAL USE

The middle of the grid carries **sixty times** the foot traffic of its corners, even when every walker picks a route at random and every route is exactly the same length.

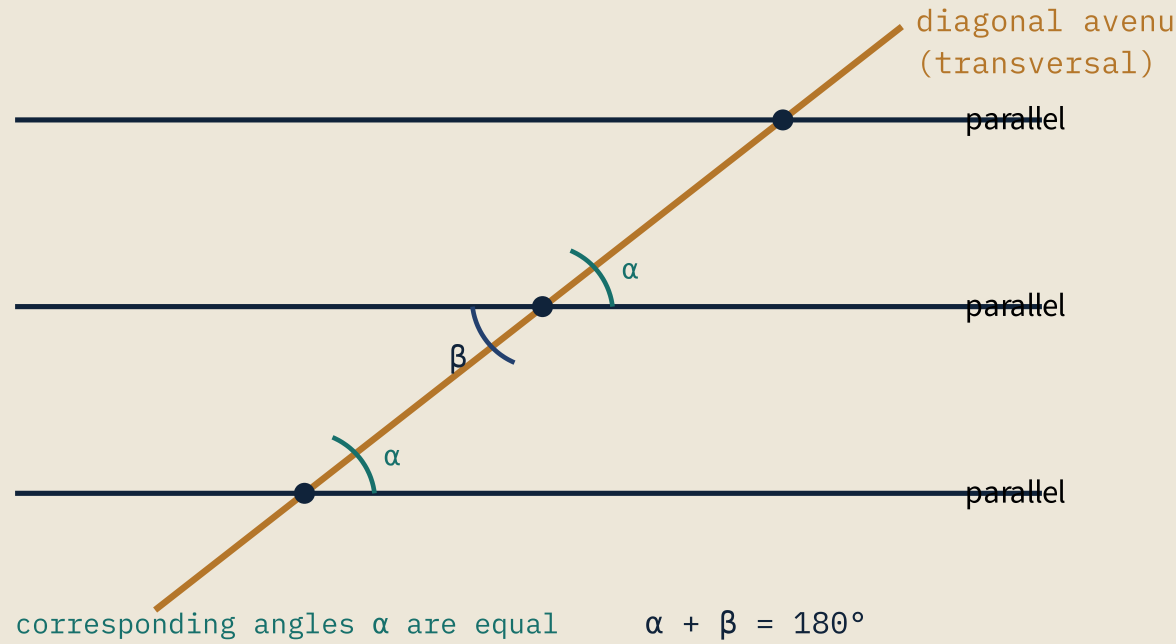
That is why downtown sidewalks are not all the same width, why the middle intersections get long crossing signals, and why the rent is higher there. The grid gives everyone freedom to choose, and the mathematics of choice funnels them through the center anyway.

It is also a sharp lesson about averages. "126 routes, 50 people" sounds like a thin crowd everywhere; in fact half of them pass through one intersection. Worth drawing out explicitly — students meet very few examples this concrete of an average concealing its distribution.

03 Transversals and diagonal avenues

VIDEO **Points and Lines, Segment 6 — transversals**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PointsLines--GoogleVoyagerSegment6.mp4

Washington, DC is a rectangular grid with diagonal avenues laid over it, and each avenue is a transversal.



One diagonal, and every intersection along it repeats the same two angles.

Corresponding angles	Same position at different intersections along the transversal. Equal when the lines are parallel.
Alternate interior angles	Opposite sides of the transversal, between the parallels. Also equal.
Co-interior angles	Same side of the transversal, between the parallels. Add to 180°.

WHAT A DIAGONAL BUYS

Take the §2 trip: 5 blocks east, 4 north. Grid distance 9 units; straight across, $\sqrt{41} \approx 6.40$ units. A diagonal avenue running that way saves 2.6 units — about **29%**.

That is the whole argument for L'Enfant's plan for Washington: the grid gives order and identical blocks, and a handful of diagonals give back most of the distance the grid costs. The price is paid at the intersections — five- and six-way junctions and sharp triangular lots. The Flatiron Building from the Architectural Prisms series sits on exactly such a corner.

04 Practice, with answers

1. A street has slope $3/5$. What is the slope of a perpendicular street? A parallel one?

ANSWER Perpendicular: $-5/2$, since $3/5 \times (-5/2) = -1$. Parallel: the same slope, $3/5$. Every street in a rectangular grid has one of these two slopes and no others.

2. B is 3 blocks east and 3 north of A. How many shortest routes, and how long is each?

> ANSWER Each is 3 + 3 = **6 blocks**; the count is $C(6, 3) = 6!/(3! \cdot 3!)$ = **20 routes**. Also readable off a Pascal grid: fill the 4 × 4 array and the far corner shows 20.

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3. Using the addition rule, find the number of routes to the intersection 3 east and 2 north of A.

> ANSWER Bottom row 1, 1, 1, 1 and left column 1, 1, 1; second row 1, 2, 3, 4; third row 1, 3, 6, **10**. Check: $C(5, 2) = 10$. ✓

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4. In the 5-east, 4-north grid, how many of the 126 routes pass through the intersection 1 east and 1 north of A?

> ANSWER A to it: $C(2, 1) = 2$. It to B: 4 east and 3 north remain, $C(7, 3) = 35$. So 2 × 35 = **70 routes** — 56% of all traffic, through an intersection one block diagonally from the start.

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5. A diagonal avenue crosses parallel streets at 62°. Find every other angle at those crossings.

> ANSWER Four angles per intersection: **62°** and its vertical angle 62°, plus **118°** twice, since 62 + 118 = 180. Because the streets are parallel, the identical set repeats at every intersection along the avenue.

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6. A diagonal avenue runs along a route 8 blocks east and 6 north. What percentage does it save?

> ANSWER Grid 14 blocks; straight line $\sqrt{(8^2 + 6^2)}$ = **10 blocks** — a doubled 3-4-5 triangle. Saving 4 of 14, about **29%** — the same percentage as the earlier trip, because 8:6 and 4:3 are the same ratio.

> BACK TO THE OPENING QUESTION
>
> If every route is the same length, why isn't every corner the same? Because equal length is not equal use. Multiply routes-in by routes-out and the middle of the grid is on 60 of the 126 routes while the far corner is on 1.
>
> Sixty to one, from a crowd choosing at random and walking the same distance either way. The wide sidewalks, the long crossing signals, and the expensive corners sit exactly where Pascal's triangle says the numbers are biggest.

Teacher's Guide · Google Earth Voyager Story: The Geometry of City Planning, Part 2
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.