The Geometry of City Planning

Taxicab distance, counting routes, and why cities are square anyway

The shortest distance between two points is a straight line. In a city you cannot walk a straight line — buildings are in the way. So what *is* the shortest distance, and how many different ways are there to walk it?

GRADE BAND	TIME	FORMAT
7–9, extensions for HS Geometry	**Two class periods**	**Video, dynamic geometry construction, path investigation**

WHAT STUDENTS WILL BE ABLE TO DO

- Distinguish straight-line distance from distance measured along streets, and calculate both.
- Show that *every* non-backtracking route through a grid is the same length — and count how many there are.
- Find the worst-case penalty a grid imposes over travelling as the crow flies.
- Compare a rectangular grid with a circular one, and find the angle at which cutting through the center beats following the ring.
- Explain, with numbers, why cities use square blocks even though a circular layout sometimes wins on distance.

STANDARDS

6.NS.C.8 points and distances in the coordinate plane · **7.G.B.4** area and circumference of a circle · **8.G.B.8** distance between two points · **HSG-GPE.B.7** perimeters and areas using coordinates · **HSG-C.B.5** arc length · **HSG-MG.A.1** and **A.3** modeling with geometry and design problems

START HERE

Open the Voyager story in Chrome and look down on downtown Houston. Have students pick two intersections a few blocks apart diagonally, trace one route with a finger, then a completely different route, and *commit to a guess* about which took longer.

Most classes guess that the zigzag is longer. Collect the guesses before §2 — being wrong first is what makes the taxicab result land.

earth.google.com/web/data=Mj8KPQo7CiExU2dRTFhjQ2VqeUFYTThGVmk3V1pVZlA1MHF3THVnZ1QSFgoUMENBOEExMDBFRDE2NzE1QzIxMUU

01 The Houston city grid

VIDEO **Points and Lines, Segment 1 — the Houston grid**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PointsLines--GoogleVoyagerSegment1.mp4

Houston was platted in 1836 by the Allen brothers: 62 blocks, 16 streets, almost every block a square **250 feet on a side**. Downtown today runs to roughly 400 blocks on that same grid.

Worth pointing out from the air: the grid is not aligned to north. The Allens turned it to line up with Buffalo Bayou, the shipping route the town depended on. A grid has to point somewhere, and the choice is usually geographic rather than cardinal — a good five-second aside that heads off the assumption that grids "should" run north–south.

A GRID IS A COORDINATE SYSTEM

Once every block is the same size, an address becomes a pair of numbers. A gridded city is easy to navigate because it hands every visitor a coordinate plane already drawn on the ground — and that is also why the mathematics below is so clean. Take one block as the unit and the city becomes graph paper.

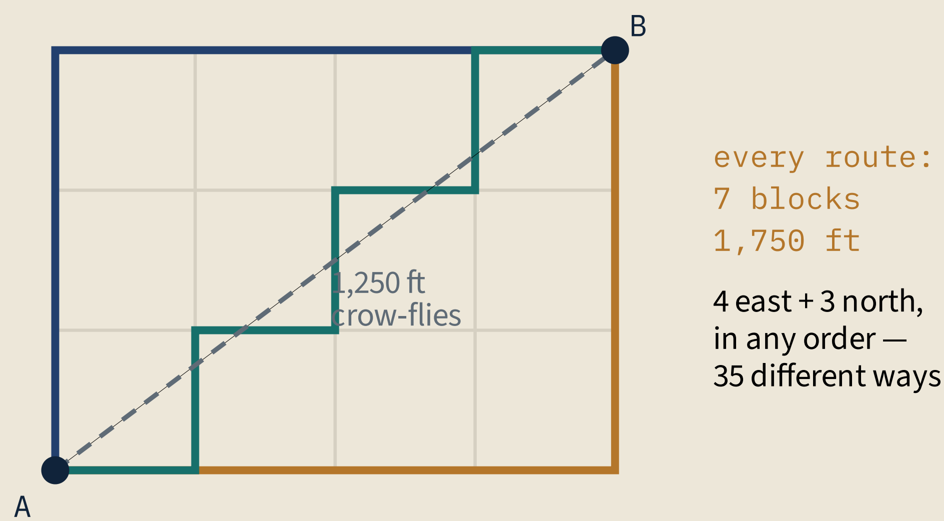
02 The shortest-distance principle

The straight line is shortest — but a street grid does not let you walk it. There are **two different distances** between two corners, and keeping them apart is the key to the whole lesson.

Straight-line distance	Crow-flies: $\sqrt{(\Delta x^2 + \Delta y^2)}$, from the Pythagorean Theorem.
Grid distance	Along streets: $\lvert\Delta x\rvert + \lvert\Delta y\rvert$. Mathematicians call it the *taxicab distance*.
Monotone route	A route that never backtracks — only east and north when the destination is east and north.

Every sensible route is the same length

If B is 4 blocks east and 3 north, any non-backtracking route contains exactly 4 east moves and 3 north moves, in some order. So every such route is **7 blocks** — 1,750 ft on Houston's blocks. The order is free; the total is not.



Three routes drawn, thirty-five available, all exactly seven blocks.

How many routes? Choose which 3 of the 7 moves go north: $C(7, 3) = \mathbf{35}$. A nice place to let combinatorics appear without announcing it.

WHAT THE GRID COSTS YOU

Straight-line distance is $\sqrt{(4^2 + 3^2)} = 5$ blocks — the 3-4-5 triangle. Grid distance is 7. The streets cost 2 extra blocks, a **40% detour**.

The ratio $(\Delta x + \Delta y)/\sqrt{(\Delta x^2 + \Delta y^2)}$ equals 1 when travel is along one street and is largest when $\Delta x = \Delta y$, where it hits $\sqrt{2} \approx \mathbf{1.414}$. So a rectangular grid never costs more than about **41%** — a bound that holds for every pair of corners in every gridded city on Earth.

VIDEO **Points and Lines, Segment 2 — hands-on with dynamic geometry**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PointsLines--GoogleVoyagerSegment2.mp4
TI-Nspire in the video; the Desmos geometry tools at desmos.com/geometry work equally well.

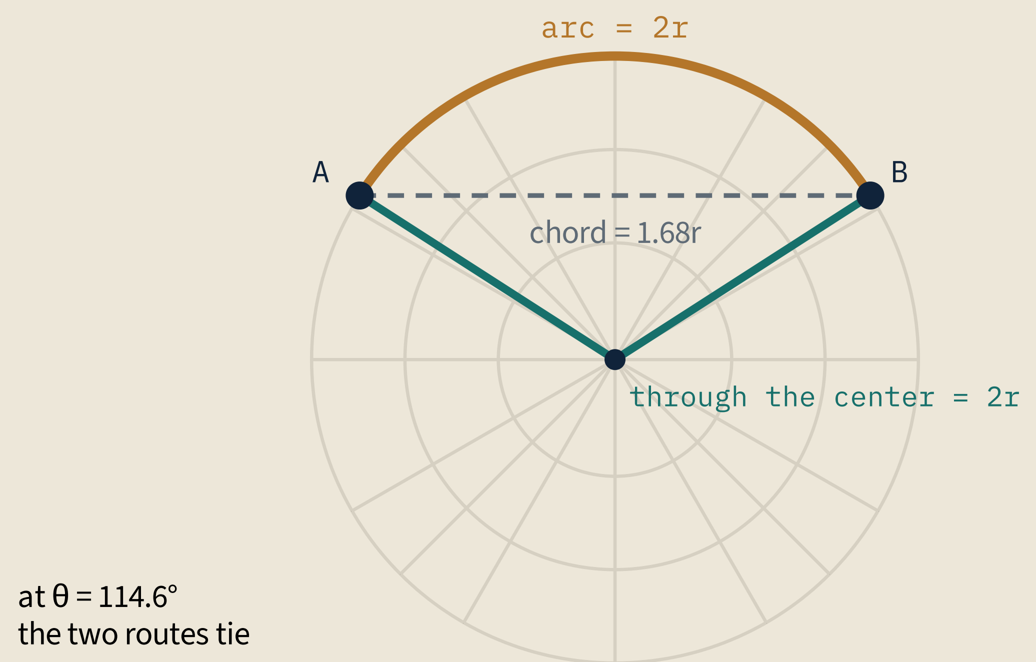
03 What about a circular grid?

Put A and B on the same ring, radius *r*, separated by central angle θ. Two routes:

- **Follow the ring:** an arc of length $r\theta$, with θ in radians.
- **Cut through the center:** in along one spoke and out along another — $2r$, whatever θ is.

The arc wins when $r\theta < 2r$. The radius cancels entirely:

$$\theta < 2 \text{ radians} \approx 114.6°$$



Below 114.6° follow the ring; above it, cut through the middle.

ANGLE θ	ARC	THROUGH CENTER	CROW-FLIES CHORD	BEST ROUTE'S DETOUR
30°	0.52r	2r	0.52r	1.2%
60°	1.05r	2r	1.00r	4.7%
90°	1.57r	2r	1.41r	11.1%
114.6°	2.00r	2r	1.68r	18.8%
150°	2.62r	2r	1.93r	3.5%
180°	3.14r	2r	2.00r	0%

AN HONEST SURPRISE — AND A CORRECTION WORTH MAKING

A circular grid's worst-case detour is **18.8%**; a rectangular grid's is **41.4%**. On distance the circular layout is not merely competitive, it is better by more than a factor of two.

The student lesson's summary says curved grids are impractical because "they require more energy to implement." That is vague, and it invites students to conclude the straight grid is simply better. It isn't — not on distance. The real argument is in §4, and it is much stronger.

04 City grids: summary

The reason is the blocks, not the routes

In a square grid every block is identical: 250 ft × 250 ft, 62,500 sq ft, at the edge of downtown exactly as at the center. One block plan, one surveying method, one way to cut lots, repeated four hundred times.

In a circular grid no two rings are alike. Rings 250 ft apart with sixteen spokes:

RING AT RADIUS	BLOCK FRONTAGE ON THE RING	BLOCK AREA
250 ft	98 ft	≈ 24,500 ft^2
1,000 ft	393 ft	≈ 98,000 ft^2
2,500 ft	982 ft	≈ 245,000 ft^2
Square grid, anywhere	250 ft	62,500 ft^2

A block ten rings out is ten times the area of one a ring in, and none is a rectangle — they are curved wedges. Every block needs its own survey, its own lots, its own building plans. Nothing repeats.

THE SAME TRADE, A THIRD TIME

Classes who have done the other Voyager stories have met this argument twice already. The Colosseum used circular arcs instead of a true ellipse because arcs need *two* stone templates instead of twenty. Triangular trusses are built equilateral because one cut and one jig make the whole frame.

City grids are square for the same reason. The winning shape is rarely the one that optimizes a single quantity — it is the one whose parts are all identical. If you teach only one idea from this series, this is a strong candidate.

One more question, for Part 2

The student lesson closes by asking why grids look more like one of two figures than the other, setting up Part 2's treatment of parallel and perpendicular lines. Here is a calculation that decides between a perpendicular grid and a slanted one.

SOMETHING TO MEASURE BEFORE ANSWERING

A block bounded by two pairs of parallel streets is a parallelogram with area:

$$\text{Area} = ab \cdot \sin \alpha$$

Since $\sin \alpha$ peaks at $\alpha = 90°$, a **perpendicular** grid encloses more land per foot of street than any slanted grid with the same block dimensions. Slant Houston's blocks to 75° and each loses 3.4% of its area; at 60°, 13.4%. Across 400 blocks that is a great deal of city thrown away for nothing — and the streets are the same length either way.

Right angles also give every corner lot the same shape and let buildings meet the street squarely.

05 Practice, with answers

1. B is 6 blocks east and 2 north of A in Houston. Find the grid distance, the crow-flies distance, and the detour.

ANSWER Grid = 8 blocks = **2,000 ft**. Crow-flies = $\sqrt{40} \approx 6.32$ blocks ≈ **1,581 ft**. Detour ≈ **26.5%** — short of the 41% worst case because the trip is mostly in one direction.

2. How many different shortest routes are there for that trip?

ANSWER Choose which 2 of 8 moves go north: $C(8, 2)$ = **28 routes**, every one 2,000 ft. Walking a different one each working day takes nearly six weeks to repeat.

3. For which trips is the grid no detour at all, and for which is it worst?

> ANSWER **No detour** when purely east–west or north–south (one of Δx, Δy is zero). **Worst** when $\Delta x = \Delta y$ — a perfect diagonal — where the ratio hits $\sqrt{2} \approx 1.414$.

---

4. Two points sit on a ring 800 m from the center, 100° apart. Which route is shorter, and by how much?

> ANSWER 100° = 1.745 rad, arc = **1,396 m**; through the center = **1,600 m**. **Follow the ring**, saving 204 m — expected, since 100° is below the crossover.

---

5. Repeat for points 140° apart on the same ring.

> ANSWER 140° = 2.443 rad, arc = 1,955 m against 1,600 m. **Cut through the middle**, saving 355 m. Above 114.6° the ring is always the long way round — which is why radial cities need both ring roads and spokes.

---

6. A developer proposes 300 ft × 200 ft blocks with streets meeting at 70°. How much land is lost per block versus perpendicular?

> ANSWER Perpendicular: 60,000 ft^2. At 70°: 60,000 × sin 70° ≈ 56,382 ft^2. A loss of about **3,600 ft^2 per block**, roughly 6% — bought for nothing, since the streets are the same length either way.

> BACK TO THE OPENING QUESTION
>
> Did the second route take longer? No — and neither would any of the other thirty-three. In a grid every non-backtracking route is exactly the same length, so choosing between them is about traffic lights, shade, and shop windows, never distance.
>
> As for whether the grid is the best possible layout: on pure distance it isn't. A circular city would send you at most 19% out of your way instead of 41%. Cities are square anyway, because a square grid is built from four hundred identical blocks and a circular one from four hundred different ones.

Teacher's Guide · Google Earth Voyager Story: The Geometry of City Planning, Part 1
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.
Media4Math® is a registered trademark. All rights reserved.