Architectural Prisms

Twists, changing cross-sections, and the prismatoid formula at One World Trade Center

Every prism in Parts 1 and 2 had the same cross-section from bottom to top. One World Trade Center does not. Slice it at the right height and you get a perfect octagon — but "the right height" is not where students will guess.

GRADE BAND	TIME	FORMAT
6–8, extensions for HS Geometry	**Two to three class periods**	**Video, net construction, scale model, cross-section investigation**

Related lessons: **Triangular Prisms** (Part 1) and **Rectangular Prisms** (Part 2). All three share a Voyager story and the §1 video. Part 3 is the capstone — §7 derives both earlier volume formulas from this one's.

WHAT STUDENTS WILL BE ABLE TO DO

- Say precisely how an antiprism differs from a prism, and count its faces, edges, and vertices.
- Find the twist angle for any antiprism, and check Euler's formula for all of them at once.
- Describe how a horizontal cross-section changes as you move up an antiprism.
- Find the exact height at which One World Trade Center's floor plan is a regular octagon — and prove it is regular.
- Use the prismatoid formula to find the volume of a solid whose cross-section is not constant.

STANDARDS

6.G.A.4 nets and surface area · **7.G.A.1** scale drawings and scale models · **7.G.A.3** two-dimensional figures that result from slicing three-dimensional figures · **7.G.B.6** area, volume, and surface area · **HSG-GMD.A.1** and **A.3** volume formulas and their justification · **HSG-MG.A.1** modeling real objects with geometric shapes

START HERE

Open the Voyager story and orbit One World Trade Center a full turn (best viewed in Chrome). Have students watch the silhouette: from one angle a straight-sided slab, 45° later a tapering obelisk.

Ask: *how can one solid have two different outlines?* Collect guesses. §6 answers it with two numbers, and the answer is more satisfying if students have committed to a wrong one first.

earth.google.com/web/data=Mj8KPQo7CiExSXd1cWVTNXpzdXNoS2l1bzNXcEJ4S2VuLU4zR1dHcVQSFgoUMDJGNDZFNUVFMzEzMTBDRTVBNTI

01 Prisms, and then antiprisms

VIDEO **What Are Prisms? — includes a discussion of antiprisms**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/WhatArePrisms.mp4
Same clip as Parts 1 and 2.

In Parts 1 and 2 the headline was that a prism has a **constant cross-section** — that is what made $V = Bh$ work. An antiprism breaks it: two congruent polygons, the top one *twisted*, joined by triangles instead of rectangles.

Antiprism	Two parallel congruent polygon bases, one rotated relative to the other, joined by a band of triangles.
Twist angle	How far the top base is rotated. For an n-gonal antiprism, $180°/n$.
Lateral face	Any face that isn't a base. Rectangles in a prism; triangles in an antiprism.
Frustum	What's left when you slice off the top parallel to the base. 1 WTC is a tapered antiprism — an antiprism frustum.
Prismatoid	Any solid whose vertices lie in two parallel planes. Prisms, pyramids, frustums, and antiprisms all qualify.

ANIMATION **Antiprism · and · Antiprism with Horizontal Cross-Section**
media4math.com/library/animated-math-clip-art-3d-geometry-antiprism
media4math.com/library/animated-math-clip-art-3d-geometry-antiprism-horizontal-cross-section
Project the cross-section animation before anything else in this section. It is the whole lesson in five seconds: in a prism the slice never changes, here it does.

02 The Freedom Tower

VIDEO **Antiprisms — One World Trade Center**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/Antiprisms.mp4

Designed by David Childs of Skidmore, Owings & Merrill and completed in 2014, the tower stands on a 200-foot square base — deliberately the footprint of each original Twin Tower. A windowless concrete podium rises about 185 feet; above that the corners chamfer back and the building becomes the antiprism.

MEASUREMENT	VALUE
Base square	200 ft
Top square (glass parapet), rotated 45°	≈ 150 ft
Parapet height	1,368 ft — the height of the original towers
Height with spire	1,776 ft
Antiprism section	≈ 1,183 ft (185 ft base to 1,368 ft parapet)

The chamfering turns four flat walls into **eight tall isosceles triangles**, alternating point-up and point-down — exactly the triangle band of an antiprism. The top square is rotated 45°, so its corners sit above the midpoints of the base's sides.

IS IT REALLY AN ANTIPRISM?

Not quite. In a true antiprism the two bases are *congruent*; here the top is 150 ft against a 200 ft base, so the tower tapers. The precise term — the one the tower's own structural engineers used — is an **elongated square antiprism frustum**.

Calling it an antiprism is a fair simplification, as calling the Flatiron a triangular prism was in Part 1. But name the simplification out loud, because the taper is exactly what makes §5 interesting.

03 Constructing an antiprism

VIDEO **Antiprism Animation · and · Net for Antiprism**
s3.amazonaws.com/Media4MathPlus/Videos/GeometryVideos/AntiprismAnimation.mp4
media4math.com/library/animated-math-clip-art-3d-geometry-net-antiprism

HANDOUT **Printable Nets (PDF)**
media4math.com/sites/default/files/library_asset/files/PrintableNets4.pdf

COUNT THE PARTS, AND COMPARE

	SQUARE PRISM	SQUARE ANTIPRISM
Faces	6	10
Edges	12	16
Vertices	8	8
Lateral faces	4 rectangles	8 triangles
Twist	0°	45°
$V - E + F$	2	2

Same eight vertices, four more edges, four more faces. In general an n-gonal antiprism has $2n$ vertices, $4n$ edges, and $2n + 2$ faces, so $2n - 4n + 2n + 2 = 2$ for *every* n at once — a nice moment to show a single algebraic check standing in for infinitely many solids.

ANTIPRISM	TWIST = 180°⁄N	VERTICES	EDGES	FACES
Triangular ($n = 3$)	60°	6	12	8
Square ($n = 4$)	45°	8	16	10
Pentagonal (n = 5)	?	?	?	?
Hexagonal (n = 6)	?	?	?	?

Blank rows: pentagonal 36°, 10, 20, 12. Hexagonal 30°, 12, 24, 14.

Build steps as given in the student version: print, cut including flaps, fold all lines, glue one side at a time, repeat, let dry.

NOTICE THE BAND

In the Part 1 and Part 2 nets the lateral faces were a straight strip of rectangles. Here they are a zigzag strip of triangles, alternating point-up and point-down. Have students count them before folding — eight for a square antiprism, $2n$ in general. That zigzag *is* the twist: it is what lets a square meet a square turned 45°.

04 A scale model of the Freedom Tower

Students build the antiprism portion using a height-to-base ratio of 6:1.

TEST THE RATIO AGAINST THE REAL BUILDING

The antiprism section runs from the top of the 185 ft podium to the 1,368 ft parapet — about 1,183 ft — on a 200 ft base:

$$1{,}183 / 200 \approx 5.9 : 1$$

Accurate to within 2%. That makes two of three lessons in this series where the supplied ratio holds up (432 Park's 15:1 was exact; the Flatiron's was about 7% off). The habit is what matters — checking takes a minute.

Note: the student lesson says "base-to-height ratio of 6:1," which reads backwards. It is height to base.

05 The slice that keeps changing

This is the new mathematical content, and the reason Part 3 is worth teaching after Parts 1 and 2.

Measure height as a fraction t of the antiprism section. Every slice is an octagon, because each of the eight triangular faces contributes one side — but the two kinds of triangle shrink in opposite directions:

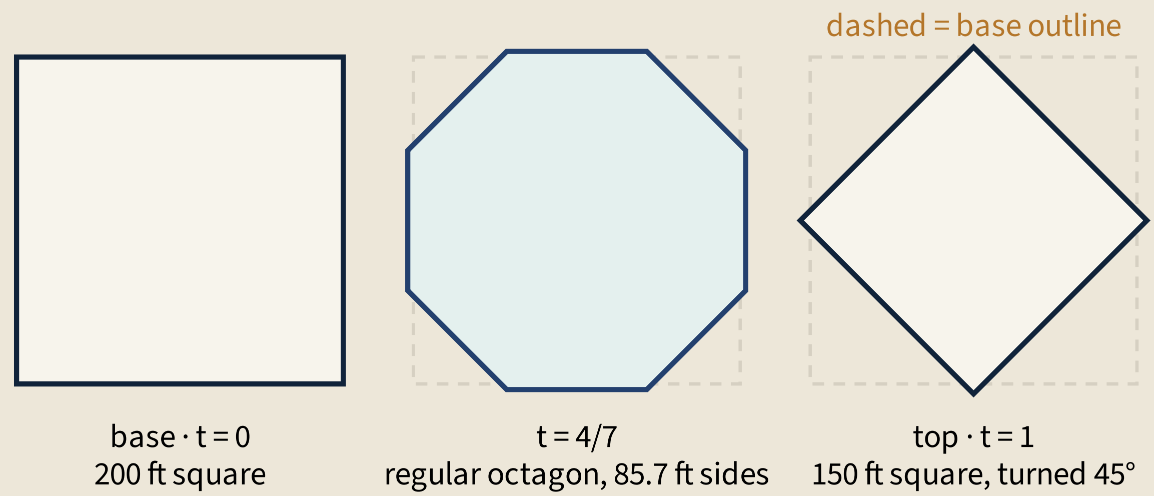
- Four point-up triangles sit on the base, so their sides shrink: $(1 - t) \times 200$.
- Four point-down triangles hang from the top, so their sides *grow*: $t \times 150$.

At the halfway point those are 100 ft and 75 ft — an octagon, but a lopsided one.

Where is the regular octagon?

$$(1 - t) \times 200 = t \times 150 \quad \rightarrow \quad t = 4/7 \approx 0.571$$

At **four sevenths** of the way up — not one half — all eight sides measure 85.7 ft. And at that same height the eight vertices land at exactly 22.5°, 67.5°, 112.5°, and so on: equal sides *and* equal angles. The cross-section is a genuinely **regular octagon**.



Three slices through the same solid. Only the middle one is regular.

THE GENERAL RESULT

For any square antiprism frustum with base side a and top side b, the cross-section is a regular octagon at exactly $t = a/(a + b)$.

Check it on a true antiprism, where $a = b$: the formula gives $t = ½$. In a genuine antiprism the regular octagon *is* at the midpoint; only the taper pushes it upward. Architects' descriptions of 1 WTC usually say the octagon appears "at its middle" — students can now be more precise than the press release.

06 Volume, and the two silhouettes

Because the cross-section changes, $V = Bh$ is unavailable. The **prismatoid formula** handles every solid in this three-part series at once:

$$V = (h/6)(A_{\text{bottom}} + 4A_{\text{middle}} + A_{\text{top}})$$

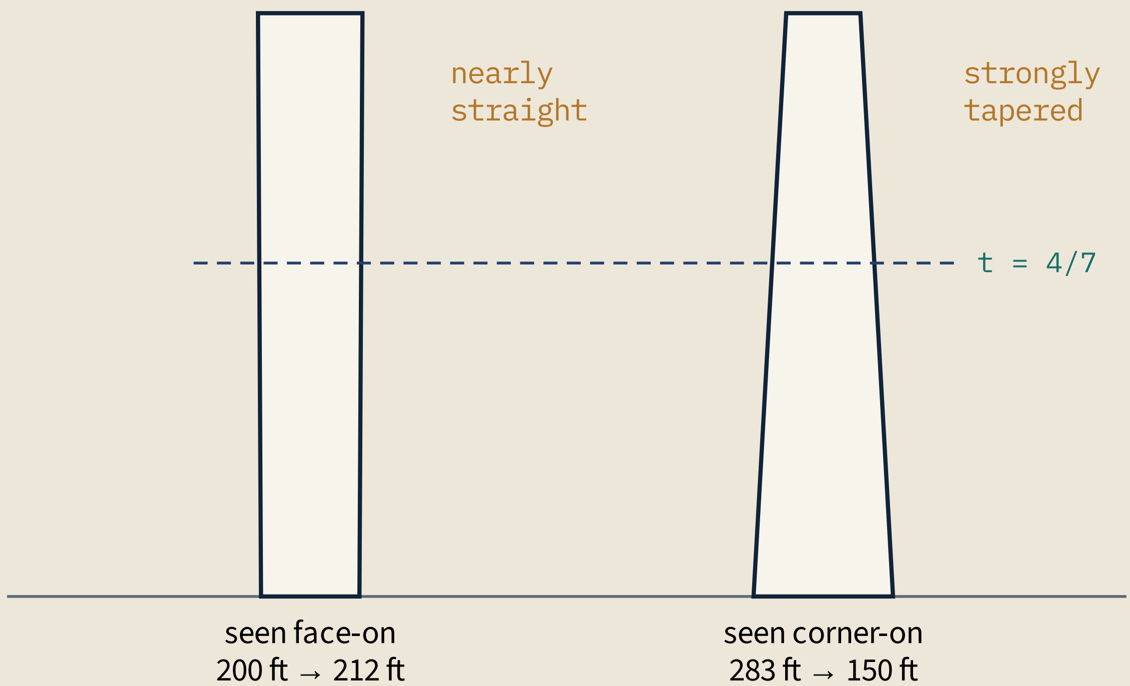
SLICE	SHAPE	AREA
Bottom	200 ft square	40,000 ft²
Middle ($t = ½$)	octagon, sides 100 and 75 ft	36,838 ft²
Top	150 ft square	22,500 ft²

$$(1{,}183/6)(40{,}000 + 147{,}353 + 22{,}500) \approx 41.4 \text{ million ft}^3$$

A plain 200 ft square prism of the same height holds 47.3 million ft^3, so the chamfering costs about 12% of the volume.

Why it looks like two different buildings

Face-on, the bottom is 200 ft across and the top presents its *diagonal*, 150 × √2 ≈ 212 ft — very slightly wider at the top, so the tower reads as a straight-sided slab. From a corner, the bottom presents its diagonal, 200 × √2 ≈ 283 ft, while the top presents a flat 150 ft side — a sharp taper.



One solid, two outlines. The 45° twist is doing all of it.

07 Practice, with answers

1. Complete the blank rows: pentagonal and hexagonal antiprisms.

ANSWER Pentagonal: twist 180/5 = **36°**, 10 vertices, 20 edges, 12 faces. Hexagonal: **30°**, 12 vertices, 24 edges, 14 faces. Both satisfy Euler: 10 − 20 + 12 = 2 and 12 − 24 + 14 = 2.

2. A square antiprism and a square prism both have 8 vertices. Why does the antiprism have more edges?

ANSWER Each prism vertex meets 3 edges: 8 × 3/2 = **12**. Each antiprism vertex meets 4 — two along its own square, two running to the other square: 8 × 4/2 = **16**. The twist means every vertex connects to *two* vertices opposite instead of one.

3. Find the two side lengths of the cross-section at $t = 1/4$, and say whether it is regular.

ANSWER Point-up sides (1 − 0.25) × 200 = **150 ft**; point-down sides 0.25 × 150 = **37.5 ft**. Very lopsided — at quarter height the plan still reads as a square with clipped corners.

4. A tower has a 120 ft base square and a 60 ft top square rotated 45°. At what fraction of its height is the cross-section a regular octagon?

ANSWER $t = 120/(120 + 60) = 2/3$. The greater the taper, the higher the regular octagon sits. Its sides measure (1 − ⅔) × 120 = 40 ft.

5. Use the prismatoid formula to check the volume of an ordinary prism.

> ANSWER All three slices are congruent, so $V = (h/6)(B + 4B + B) = \boldsymbol{Bh}$. The Part 1 and Part 2 formula falls straight out — it was the special case all along.

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6. Try the same check on a pyramid, where the top shrinks to a point.

> ANSWER $A_{top} = 0$ and the middle slice is a half-scale copy, so $A_{middle} = B/4$. Then $V = (h/6)(B + B + 0) = \boldsymbol{\tfrac{1}{3}Bh}$ — the pyramid formula, and the ⅓ that opened this whole Voyager series. Worth doing on the board even if you skip the rest of the practice set.

> BACK TO THE OPENING QUESTION
>
> How can one solid have two outlines? The 45° twist puts a flat side and a diagonal on opposite ends of the building. Face-on: a 200 ft side at the bottom, a 212 ft diagonal at the top, so it looks straight. Corner-on: a 283 ft diagonal at the bottom, a 150 ft side at the top, so it looks like an obelisk.
>
> The engineers had a reason beyond appearance. A tower with a constant cross-section sheds wind vortices at a steady rhythm, and a steady rhythm can drive a building toward resonance. A cross-section that is a different shape at every height never lets that rhythm establish itself. The twist that makes One World Trade Center interesting to slice is the same twist that keeps it still in the wind.

Teacher's Guide · Google Earth Voyager Story: Architectural Prisms, Part 3
Companion to the online lesson. Videos, animations, nets, and the Voyager story are linked by URL above; the interactive version carries them inline.