Architectural Prisms

Slenderness, surface area, and the cost of being thin at 432 Park Avenue

432 Park Avenue is 1,396 feet tall and 93 feet wide. That is fifteen times taller than it is wide — a pencil standing on a city block. Every consequence of that shape is something students can calculate.

GRADE BAND	TIME	FORMAT
6–8, extensions for HS Geometry	**Two class periods**	**Video, net construction, scale model, ratio investigation**

Related lessons in this series: **Triangular Prisms** (Part 1) and **Antiprisms** (Part 3), both in the Media4Math library. Part 1 and Part 2 share a Voyager story and a video (§1), so they can be taught back to back or independently.

WHAT STUDENTS WILL BE ABLE TO DO

- Find the volume and surface area of a rectangular prism with a square base.
- Compute a slenderness ratio and use it to compare buildings that differ wildly in size.
- Explain why a tall thin building has far more exterior surface than a compact one holding the same volume.
- Fold a net into a square prism and check the count of faces, edges, and vertices.
- Judge how well a geometric model matches a real building, and say exactly where it stops matching.

STANDARDS

6.RP.A.1 and **A.3** ratio reasoning · **6.G.A.4** nets and surface area · **7.G.A.1** scale drawings and scale models · **7.G.B.6** area, volume, and surface area of three-dimensional objects · **HSG-GMD.A.3** using volume formulas · **HSG-MG.A.1** and **A.3** modeling with geometry and solving design problems

START HERE

Open the Voyager story and press Present (best viewed in Chrome). Have students orbit 432 Park Avenue — north, east, south, west — and notice it looks identical from every direction.

Then ask: *why would an architect choose a square?* A rectangle would give bigger apartments on two sides, so something about the square must be worth more. Collect ideas and leave them up; the closing panel answers with wind loading and the perimeter-minimizing property of the square.

earth.google.com/web/data=Mj8KPQo7CiExSXd1cWVTNXpzdXNoS2l1bzNXcEJ4S2VuLU4zR1dHcVQSFgoUMDJGNDZFNUVFMzEzMTBDRTVBNTI

01 What makes a prism a prism

VIDEO **What Are Prisms?**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/WhatArePrisms.mp4
Same clip as Part 1 — skip it if students have just seen it.

Prism	A solid with two congruent, parallel faces (the bases) joined by parallelogram faces.
Cross-section	The shape produced by slicing the solid. In a prism, every slice parallel to the base is congruent to the base.
Right prism	Lateral edges perpendicular to the bases — it stands straight up.
Slenderness ratio	Height of a building divided by the width of its base. Engineers call a tower "slender" from about 10:1.
Aspect ratio	The general term for a comparison of two dimensions of the same object.

$$V = s^2 h \quad \mathrm{SA} = 2s^2 + 4sh$$

02 432 Park Avenue

VIDEO **Rectangular Prisms — 432 Park Avenue**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/RectangularPrisms.mp4

Designed by Rafael Viñoly with structural engineers WSP Cantor Seinuk and completed in 2015, the tower rises 1,396 feet on a footprint just 93 feet square. Viñoly reduced the whole building to one idea — the square — repeated until it reached the sky. No setbacks, no taper, no curtain wall hiding the frame.

MEASUREMENT	VALUE
Side of the square base	93 ft
Height	1,396 ft (425.5 m)
Floors	96 nominal, 85 structural
Completed	2015

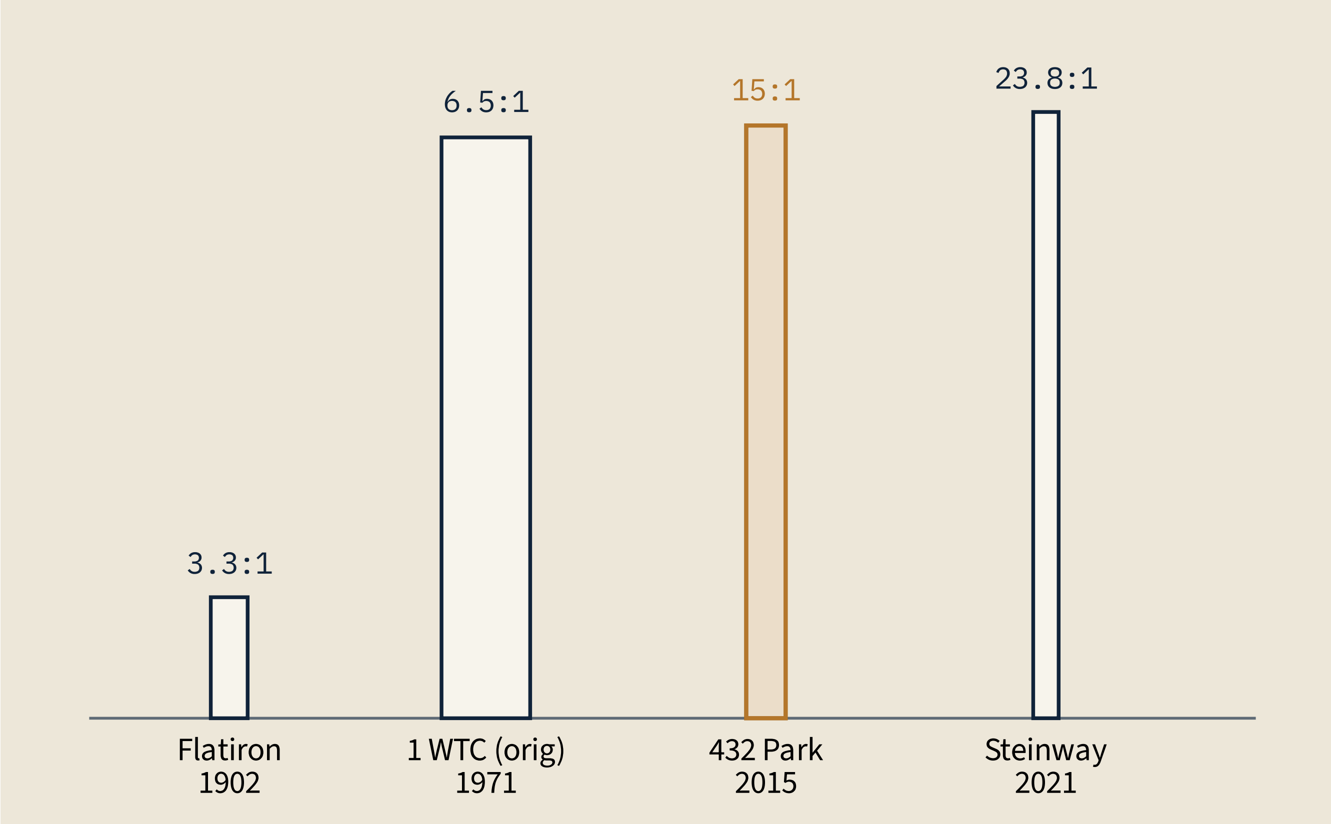
The number that defines the building

$$1{,}396 / 93 \approx 15.0$$

Engineers start calling a tower slender at about 10:1. At 15:1, 432 Park was near the frontier of what was buildable when it went up. The ratio also lets students compare buildings of wildly different sizes on one scale — including the Flatiron from Part 1.

BUILDING	HEIGHT	BASE WIDTH	SLENDERNESS
Flatiron Building (1902)	285 ft	87 ft	3.3 : 1
1 WTC, original North Tower (1971)	1,368 ft	209 ft	6.5 : 1
432 Park Avenue (2015)	1,396 ft	93 ft	?
Steinway Tower, 111 W 57th (2021)	1,428 ft	60 ft	23.8 : 1

Blank row: 1,396/93 = 15.01, so 15:1.



Same scale, real proportions. The Flatiron and 1 WTC are nearly the same width; the towers beside them are not.

The original North Tower of the World Trade Center was almost exactly as tall as 432 Park — 1,368 ft against 1,396 — on a base 209 feet square. Same height, more than twice the width. That comparison is what "slender" actually means, and it lands better than any definition.

03 Constructing a rectangular prism

VIDEO **Rectangular Prism Animation · and · Net for a Rectangular Prism**
s3.amazonaws.com/Media4MathPlus/Videos/GeometryVideos/RectangularPrismAnimation.mp4
s3.amazonaws.com/Media4MathPlus/Videos/GeometryVideos/RectangularPrismNet.mp4

HANDOUT **Printable Nets (PDF)**
media4math.com/sites/default/files/library_asset/files/PrintableNets3.pdf

COUNT THE PARTS

A square prism has **6 faces**, **12 edges**, and **8 vertices** — the same counts as a cube, because a cube *is* a square prism with all edges equal. Euler's formula still holds: 8 − 12 + 6 = **2**.

Stretching a cube into a 15:1 tower changes every length and every area but not a single count. Worth saying out loud — it separates measurement from topology, and students rarely meet that distinction anywhere else.

Build steps, as given in the student version: print the net; cut along the outline including flaps; fold along all lines; glue one side at a time, attaching flaps to the underside of the opposite side; repeat; let the glue dry before handling.

04 A scale model of 432 Park Avenue

Students build the model from a single ratio: tower height to square side is 15:1. The net image in the student version shows the target shape; flaps make gluing easier.

TEST THE RATIO AGAINST THE REAL BUILDING

The model says 15:1. The real tower is 1,396/93 = **15.01**. That is as close as a classroom ratio ever gets — not an approximation but the actual proportion.

Worth pausing on if the class built the Flatiron model in Part 1, where the given height ratio ran about 7% off. Same series, same kind of instruction, very different accuracy. The transferable habit is checking a supplied ratio rather than assuming it: sometimes it is exact, sometimes it is merely convenient.

Where the prism model stops working

432 Park is not solid all the way up. Every twelfth floor a two-story slot is left completely open — no walls, no glass — so wind blows through the building instead of pushing against it. Five of these voids cut the tower's sway by up to 30%.

Estimating what they cost in volume is a good modeling exercise because it forces students to derive a figure nobody gave them:

$$1{,}396/96 \approx 14.5 \text{ ft per floor}$$

Five voids × two floors × 14.5 ft ≈ 145 ft of open height, about **10%** of the tower. So an honest volume estimate is roughly 90% of what s^2h predicts.

HOW GOOD IS THE MODEL? — THE PAYOFF OF THE TWO-LESSON PAIR

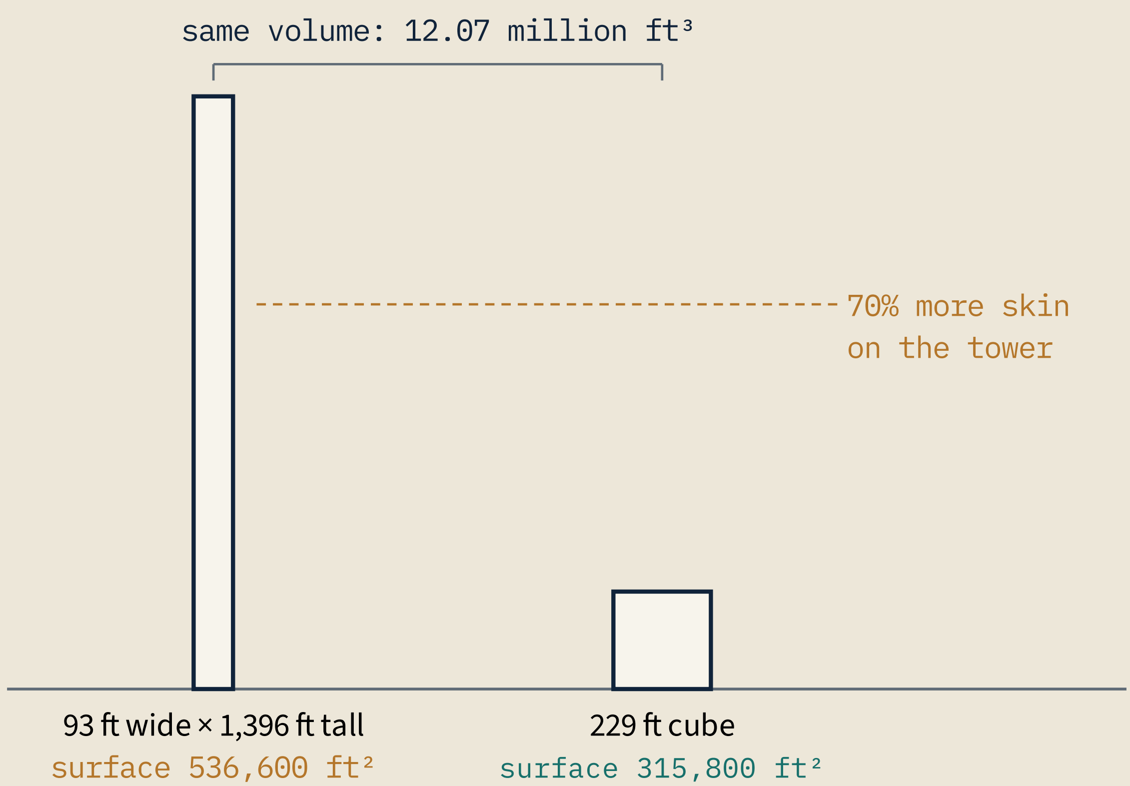
The published floor plate for 432 Park is about 8,740 sq ft. The square gives $93^2 = 8{,}649$ sq ft — within **1%**.

In Part 1 the Flatiron's published floor area came in more than 50% *above* the prism model. Neither model was wrong. The Flatiron's extra space sat in basements and a ground-floor extension outside the triangular prism; 432 Park really is the shape it looks like. Predicting which buildings a model will fit is most of what modeling is, and these two lessons together make that point better than either does alone.

05 The price of being thin

This section is new material, and it is where the slenderness ratio earns its keep. A building's floors are what you sell; its exterior surface is what you build, clad, insulate, heat, cool, and wash. Being thin wrecks the relationship between the two.

432 Park encloses $93^2 \times 1{,}396 \approx 12.07$ million cubic feet. A cube of the same volume would have side $\sqrt[3]{12{,}074{,}004} \approx 229$ ft.



Identical volume. The tall thin version has to build, clad, and heat far more wall.

Cube: $6 \times 229^2 \approx 315{,}800$ sq ft. Tower: $2(93^2) + 4(93)(1{,}396) \approx 536{,}600$ sq ft. Same volume inside, **70% more surface** outside.

That extra skin is not a design flaw — it is the price of the views and the address. But it is a real number, and it explains why supertall slender towers cost so much more per square foot than ordinary buildings of the same size.

06 Practice, with answers

1. Find the volume of 432 Park Avenue, treating it as a solid square prism.

> ANSWER $V = s^2h = 93^2 \times 1{,}396 = 8{,}649 \times 1{,}396 =$ **12,074,004 cu ft**, about 342,000 m^3.

2. Find its total surface area, then the area of just the four walls.

> ANSWER Walls only $= 4sh = 4 \times 93 \times 1{,}396 =$ **519,312 sq ft**. Adding the two 8,649 sq ft squares gives **536,610 sq ft**. The walls are 97% of the total — in a 15:1 tower the top and bottom barely register.

3. Fill in the blank row: what is 432 Park's slenderness ratio?

> ANSWER 1,396/93 = **15.01**, so 15:1 — more than four times as slender as the Flatiron, more than twice the original 1 WTC, still well short of the Steinway Tower's 23.8:1.

4. Using the void estimate in §4, give a better volume figure for the tower.

> ANSWER About 10.4% of the height is open, so roughly $0.896 \times 12{,}074{,}004 \approx$ **10.8 million cu ft**. Require students to state the assumption alongside the answer: five voids, two floors each, at an average 14.5 ft per floor.

5. A tower has a 60 ft square base and a slenderness ratio of 20:1. How tall is it, and how much wall does it have?

> ANSWER Height = 20 × 60 = **1,200 ft**. Lateral area = 4 × 60 × 1,200 = **288,000 sq ft**. Volume = 60^2 × 1,200 = 4,320,000 cu ft — about a third of 432 Park's, from a building nearly as tall.

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6. Among all rectangles with an area of 8,649 sq ft, which has the shortest perimeter?

> ANSWER The **square**, 93 × 93, perimeter 372 ft. A 60 × 144.15 rectangle has the same area but a perimeter of 408 ft — nearly 10% more wall for identical floor space. This is the mathematical half of the answer to the opening question.

BACK TO THE OPENING QUESTION

Why a square? Two reasons, both mathematical.

Wind. A square presents the same 93-foot face to a gust from any direction, so the engineers had one wind problem instead of two. A long thin rectangle would be far stiffer along one axis than the other.

Perimeter. Among all rectangles enclosing a given floor area, the square has the shortest perimeter — the least façade to build, clad, and heat per square foot sold. For a tower already carrying 70% more skin than a compact building of the same volume, that saving is the difference between a buildable design and an unbuildable one.

Teacher's Guide · Google Earth Voyager Story: Architectural Prisms, Part 2
Companion to the online lesson. Videos, nets, and the Voyager story are linked by URL above; the interactive version carries them inline.