TEACHER'S GUIDE · GOOGLE EARTH VOYAGER STORY

Triangular Structures

Rigidity, congruence, and similarity in the Bank of China and Eiffel Towers

A skyscraper in Hong Kong and a tower in Paris, built a century apart from different materials for different reasons. Both are made almost entirely of triangles. That is not a style choice — it's the one polygon that can't change shape.

GRADE BAND	TIME	FORMAT
9–10 · High School Geometry	**Three to four class periods**	**Video, construction activity, hands-on build, proof**

WHAT STUDENTS WILL BE ABLE TO DO

- Explain why a triangle is rigid and a quadrilateral isn't, using the SSS Postulate rather than intuition.
- Construct an isosceles triangle on the TI-Nspire, in Desmos, or in GeoGebra.
- Use the triangle inequality to decide whether a set of member lengths can build a truss at all.
- Find the height, apex angle, and load geometry of an isosceles truss with a vertical support.
- Prove two triangles similar using AA, and apply similarity to real measurements taken off a building.

STANDARDS

7.G.A.2 conditions that determine a unique triangle *(prior knowledge)* · **HSG-CO.B.7** and **B.8** congruence criteria including SSS · **HSG-CO.C.10** theorems about triangles · **HSG-SRT.A.2** and **A.3** similarity and the AA criterion · **HSG-SRT.B.5** using congruence and similarity to prove relationships · **HSG-SRT.C.8** right triangles and the Pythagorean Theorem · **HSG-MG.A.1** and **A.3** modeling with geometry and solving design problems

START HERE

Open the Voyager story and view the Bank of China Tower from several angles (best viewed in Chrome), then the Eiffel Tower. Have students count triangles until they lose track, then pose the question: *why not squares?* Steel and iron come in straight pieces and a square frame is simpler to build, so something must make the extra trouble worth it.

Collect predictions before §5. Students who have guessed wrong about rigidity remember the SSS argument far better than students who were told it first.

earth.google.com/web/data=Mj8KPQo7CiExcjl2Y2FjbllUOWFhMFp2N3VGekpfeTdWVnFUb1dPeVkSFgoUMDA1REM4NjEwNTBFQUVEN0E4NUQ

01 The Bank of China Tower

VIDEO **Introduction — triangles in the Bank of China Tower**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/Introduction.mp4

I. M. Pei's tower, completed in 1990, rises 315 m to the roof and 367.4 m to the tops of its masts. It was the first supertall building outside the United States, and the first high-rise anywhere built as a composite **space frame** — a three-dimensional truss doing the job a concrete core usually does.

The geometry starts with one square. Pei drew a 52 m square floor plan and cut it along both diagonals, producing four quadrants. Each quadrant is a triangular prism, and the four rise to different heights — the north one stops around floor 17, the east and west higher, and the south prism continues alone to the top. Diagonal braces carry load down the outside to four corner columns.

WORK OUT THE QUADRANT

Have students draw the 52 m square with both diagonals and classify a quadrant, then find its sides and area.

Answer: the diagonals of a square meet at right angles and bisect each other, so each quadrant is a right isosceles triangle — two sides of $52/\sqrt{2} \approx 36.8$ *m meeting at 90°, with the 52 m building edge as hypotenuse. Area ≈ 676* m^2*, and* $4 \times 676 = 2{,}704 = 52^2$.

02 The geometry of the Eiffel Tower

VIDEO **Eiffel Tower, Part 1 — the tower, its designers, and triangular trusses**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart1.mp4

The tower was designed in Gustave Eiffel's office by engineers Maurice Koechlin and Émile Nouguier with architect Stephen Sauvestre, and built between 1887 and 1889 for the Paris World's Fair. It stands on a square base 125 m on a side, reaches 300 m of ironwork (330 m today with antennas), and is assembled from roughly 18,000 iron pieces joined by about 2.5 million rivets. The iron weighs 7,300 tonnes.

The video introduces a set of engineering terms but does not define them. They are collected here because the rest of the lesson leans on them constantly — worth putting on the board before playing the clip.

Truss	A rigid framework built from straight members arranged in triangles.
Member	One straight piece of the framework — a single beam or bar.
Vertex	A point where members meet. In a real truss, a joint.
Force	A push or pull on the structure, measured in newtons or pounds.
Compression	A force squeezing a member from both ends, trying to shorten it.
Tension	A force pulling a member from both ends, trying to stretch it.
Static equilibrium	All forces cancel, so nothing accelerates. A standing building is in static equilibrium; a falling one is not.

EXPLORE **Google Earth view — first two panels**
earth.google.com/web/data=Mj8KPQo7CiExcjl2Y2FjbllUOWFhMFp2N3VGekpfeTdWVnFUb1dPeVkSFgoUMDA1REM4NjEwNTBFQUVEN0E4NUQ
Keep this window open — the lesson returns to it at the end.

03 Constructing an isosceles triangle with the TI-Nspire

VIDEO **Eiffel Tower, Part 2 — the construction, keystroke by keystroke**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart2.mp4

The construction uses **vertex**, **perpendicular bisector**, **line segment**, **base**, **endpoint**, **side**, **angle**, **length**, and **congruent sides**.

WHY THE CONSTRUCTION WORKS

Draw base *AB*, construct its perpendicular bisector, put apex *C* anywhere on that line. The triangle is isosceles every time — and not by luck. Every point on the perpendicular bisector of a segment is *equidistant from the segment's endpoints*, so $CA = CB$ automatically.

Have students drag *C* and watch the two measured lengths change together and stay equal. A construction that survives dragging is a proof students can see — make that point explicitly, because many will otherwise read the software as merely drawing accurately.

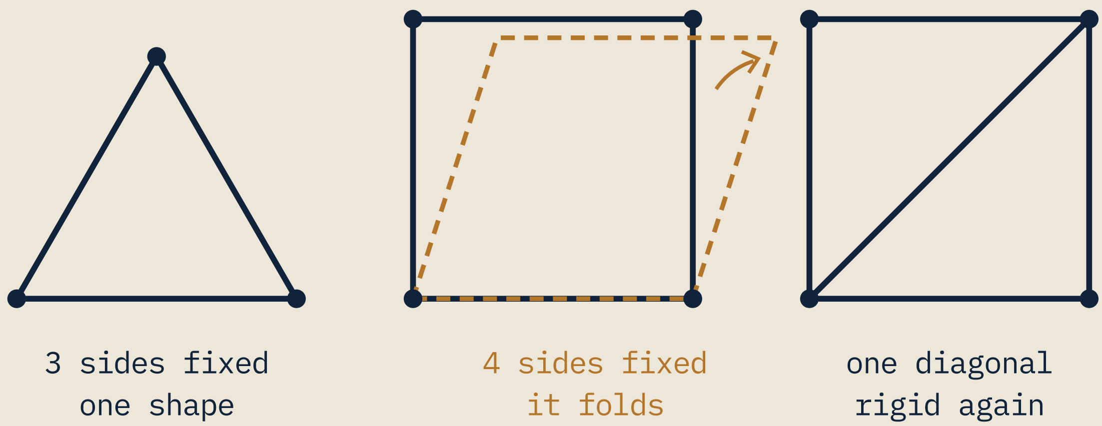
04 The same construction in Desmos and GeoGebra

VIDEO **Isosceles triangle in Desmos Geometry · and · GeoGebra Geometry**
s3.amazonaws.com/Media4MathPlus/Videos/DesmosVideos/DesmosVideo--IsoscelesTriangle.mp4
s3.amazonaws.com/Media4MathPlus/Videos/DesmosVideos/GeogebraVideo--IsoscelesTriangle.mp4

Different tools, identical construction. Worth saying aloud: the perpendicular bisector method is not a feature of any one program, it is a theorem. The software only draws what the geometry already guarantees.

05 The rigidity of triangles

VIDEO **Eiffel Tower, Part 3 — rigidity, and the flexibility of quadrilaterals**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart3.mp4



Same joints, same members. Only the middle frame can move.

Build all three from card strips and paper fasteners and push sideways on the top. The triangle resists; the square folds into a parallelogram without any member changing length. Add one diagonal and the folding stops, because the diagonal turns the square into two triangles.

That is the principle behind every truss: **triangulate the frame.** An n-sided polygon needs $n - 3$ diagonals to break into triangles, producing $n - 2$ of them.

FRAME	SIDES	DIAGONALS NEEDED	TRIANGLES FORMED
Triangle	3	0	1
Quadrilateral	4	1	2
Pentagon	5	2	3
Hexagon	6	?	?
Octagon	8	?	?
n-gon	n	$n - 3$	$n - 2$

Blank rows: hexagon 3 diagonals and 4 triangles; octagon 5 diagonals and 6 triangles.

06 The Side-Side-Side Postulate

VIDEO **Eiffel Tower, Part 4 — the SSS Postulate**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart4.mp4

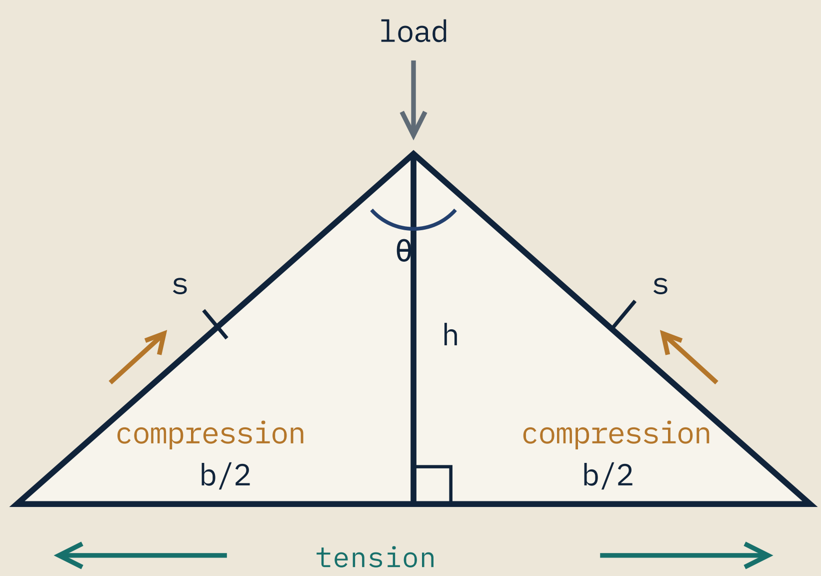
RIGIDITY AND SSS ARE THE SAME STATEMENT

SSS says three side lengths determine a triangle completely — exactly one triangle has those sides, up to rigid motion. The angles come along for free, so the shape cannot change unless a side changes length. A steel member doesn't change length, therefore a triangular frame cannot change shape. Rigidity is a theorem, not a property of steel.

Then ask for the quadrilateral version. There isn't one: no "SSSS" postulate exists, because four side lengths don't determine a quadrilateral — a square and a squashed rhombus have identical sides. The missing postulate is exactly why the middle frame folds. This is the intellectual centre of the lesson; give it time.

07 Triangular trusses

VIDEO **Eiffel Tower, Part 5 — isosceles trusses and the vertical support**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart5.mp4



The vertical support makes two congruent right triangles. Legs squeeze, bottom chord stretches.

Push down on the apex and the legs are driven toward each other — **compression**. Their feet try to slide apart, so the bottom chord is stretched — **tension**. Every member of a loaded truss is doing one job or the other.

ONE SEGMENT, FOUR JOBS

In an isosceles triangle the support from apex to base is simultaneously the **perpendicular bisector** of the base, the **altitude**, the **median**, and the **angle bisector** of the apex angle. In a scalene triangle those are four different segments — a comparison worth drawing on the board, since the coincidence is what makes isosceles trusses convenient.

$$h = \sqrt{s^2 - (b/2)^2}$$

08 Triangle inequalities and truss size

VIDEO **Eiffel Tower, Part 6 — the triangle inequality and truss limits**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart6.mp4

The **triangle inequality** says the sum of any two sides must exceed the third. For an isosceles truss with legs s and base b, the binding condition is $b < 2s$.

Investigation: how wide can a truss get?

Fix the legs at s = 10 m and widen the base.

BASE B (M)	HEIGHT H (M)	APEX ANGLE θ	WHAT IT LOOKS LIKE
2	9.95	11.5°	a narrow spike
6	9.54	34.9°	tall and slim
10	8.66	60.0°	equilateral
14	?	?	?
18	4.36	128.3°	wide and flat
19.9	1.00	168.5°	almost a straight line
20	0	180°	not a triangle at all

Blank row: $b = 14$ gives $h = \sqrt{51} \approx 7.14$ m and $\theta \approx 88.9°$.

At b = 20 the legs lie flat along the base and the triangle has collapsed to a segment. The triangle inequality is not an abstract rule about numbers — it marks the exact width at which a truss stops being a structure.

THINK IT THROUGH

Ask why the height collapses so much faster than the base grows — 8.66 m down to 1.00 m while the base goes 10 m to 19.9 m. The height depends on $s^2 - (b/2)^2$, and that difference shrinks toward zero far faster than b itself changes. Students who see this also understand why near-degenerate triangles are numerically unstable.

09 Equilateral triangles

VIDEO **Eiffel Tower, Part 7 — equilateral triangles in trusses**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart7.mp4

The equilateral triangle is the middle row of the table above: $b = s = 10$ m, apex angle exactly 60°, height $5\sqrt{3} \approx 8.66$ m. A useful balance point — wide enough to span, tall enough to be stiff, far from collapse at $b = 2s$.

It has a manufacturing advantage too. Every member is the same length and every joint the same 60° angle, so one cut, one jig, and one connector type build the whole frame. All equilateral triangles are similar to one another, so a truss scales up or down without redesigning any angle.

10 Similar triangles

VIDEO **Eiffel Tower, Part 8 — similar triangles and proportional parts**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart8.mp4

Above the second platform the tower's four legs have merged into a single tapering pylon — geometrically, a truncated pyramid. Two measurements describe it:

LEVEL	HEIGHT ABOVE GROUND	WIDTH OF THE PYLON
Second platform	115.73 m	31.70 m
Summit platform	276.13 m	10.00 m

The sides are straight, so the width shrinks at a constant rate of (31.70 − 10.00)/(276.13 − 115.73) ≈ 0.1353 m per metre of height.

Where would the tower come to a point?

Extend the sloping sides upward until they meet. From the summit you need 10.00/0.1353 ≈ 73.9 m more height for the width to reach zero — an apex at about **350 m**.

The real tower stops at 300 m. It never reaches its own apex; the top is cut off 50 m short. That is what "truncated pyramid" means, and why the summit is a platform rather than a spike.

> CHECK THE SIMILARITY
>
> Half-widths: 5.00/15.85 = **0.3155**. Distances below the virtual apex: 73.87/234.27 = **0.3154**. The same ratio from two completely different measurements — similarity doing what it promises, and a check that the straight-taper model matches the real building.

11 Geometric proof

VIDEO **Eiffel Tower, Part 9 — proving adjacent trusses similar**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart9.mp4

Let A be the virtual apex of the pylon, BC the horizontal chord at one platform, and DE the chord at a higher one.

Given: $DE \parallel BC$, with D on AB and E on AC. **Prove:** $\triangle ADE \sim \triangle ABC$.

STATEMENT	REASON
1. $DE \parallel BC$	Given — the tower's horizontal chords are built parallel
2. $\angle ADE \cong \angle ABC$	Corresponding angles, parallel lines cut by transversal AB
3. $\angle A \cong \angle A$	Reflexive property of congruence
4. $\triangle ADE \sim \triangle ABC$	AA similarity criterion

Two angles were enough, and no length was ever measured — which is precisely why similarity is the right tool for a structure nobody can put a tape measure around.

12 Conclusion

VIDEO **Eiffel Tower, Part 10 — the tower's place in France and world culture**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/EiffelTowerPart10.mp4

EXPLORE **Google Earth view — remaining panels**
earth.google.com/web/data=Mj8KPQo7CiExcjl2Y2FjbllUOWFhMFp2N3VGekpfeTdWVnFUb1dPeVkSFgoUMDA1REM4NjEwNTBFQUVEN0E4NUQ
Other architectural forms that use triangles.

13 Practice, with answers

1. The Bank of China Tower's floor plan is a 52 m square cut by both diagonals. Classify one quadrant and find its side lengths and area.

ANSWER A **right isosceles triangle**. The diagonals bisect each other at right angles, so the half-diagonals are the legs: $52/\sqrt{2} \approx 36.8$ m each meeting at 90°, with a 52 m hypotenuse. Area ≈ 676 m^2; four quadrants give 2,704 $m^2 = 52^2$.

2. An isosceles truss has legs of 13 m and a base of 10 m. Find the height of the vertical support and the apex angle.

ANSWER $h = \sqrt{(13^2 - 5^2)} = \sqrt{144} =$ **12 m** — a 5-12-13 right triangle on each side. Apex angle = $2 \cdot \arcsin(5/13) \approx$ **45.2°**.

3. Can a truss be built from members of 4 m, 5 m, and 10 m?

ANSWER **No.** $4 + 5 = 9 < 10$, so the triangle inequality fails. The two shorter members cannot reach across the longest one at any angle.

4. Using the taper rate from §10, how wide is the Eiffel Tower's pylon at 200 m above the ground?

ANSWER $31.70 - 0.1353 \times (200 - 115.73) = 31.70 - 11.40 \approx$ **20.3 m**. At the 300 m top of the ironwork the same formula gives about 6.8 m.

5. How many diagonals brace a hexagonal frame, and an octagonal one?

ANSWER Hexagon: $6 - 3 =$ **3** diagonals, 4 triangles. Octagon: $8 - 3 =$ **5** diagonals, 6 triangles. Every extra side adds one diagonal and one triangle.

6. Fill in the missing truss row: legs 10 m, base 14 m.

ANSWER $h = \sqrt{(100 - 49)} = \sqrt{51} \approx$ **7.14 m**; apex angle $2 \cdot \arcsin(0.7) \approx$ **88.9°** — just under a right angle, noticeably flatter than the equilateral case.

BACK TO THE OPENING QUESTION

Why not squares? Because a square frame with fixed side lengths still has a choice about its shape, and a triangular one doesn't. SSS guarantees three side lengths pin down every angle, so a triangle made of members that can't stretch is a triangle that can't move. There is no matching postulate for quadrilaterals, and that missing theorem is the reason a square frame folds flat under a sideways push.

Eiffel triangulated his iron in 1889. Pei and his engineer Leslie Robertson triangulated their steel in 1990. A century apart, in different materials, on opposite sides of the world — the same theorem, holding both buildings up.

Teacher's Guide · Google Earth Voyager Story: Triangular Structures
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.