The Geometry of Sustainable Architecture

Size versus shape, why the cube wins, and why nobody builds one

A building loses heat through its skin and holds heat in its volume. The ratio of the two decides how hard it is to keep warm — and that ratio is pure geometry, decided before a single window is chosen.

GRADE BAND	TIME	FORMAT
9–12 · Algebra 2 through Precalculus	**Two class periods**	**Video, algebraic modeling, Desmos investigation**

WHAT STUDENTS WILL BE ABLE TO DO

- Write the surface area and volume of a square prism in terms of one variable and a shape parameter.
- Separate the two things that change the ratio: the building's *size* and its *shape*.
- Explain why the ratio falls as a building gets bigger, whatever its shape.
- Find the shape that minimizes surface area at fixed volume — and prove it is the cube.
- Explain why real towers are built nowhere near that optimum.

STANDARDS

HSA-CED.A.2 creating equations in two or more variables · **HSA-SSE.B.3** rewriting expressions to reveal properties · **HSF-IF.B.4** and **C.7d** interpreting and graphing rational functions · **HSG-GMD.A.3** using volume formulas · **HSG-MG.A.1**, **A.2** and **A.3** modeling, per-unit reasoning, and design problems

START HERE

Open the Voyager story in Chrome and look at the Citigroup Center in Manhattan. Then pose the question and make students commit before §3:

You must build one million cubic metres. Tall and thin, or short and squat — which loses less heat, and is that what anyone actually builds? Classes almost always answer "tall and thin," partly because the Desmos activity as written points that way. §3 shows why that reading is wrong.

earth.google.com/web/data=Mj8KPQo7CiExRGs1N3VKZUpsZVVRVHlfcTk5eTZ6UzllbmhnX080emISFgoUMEE4RDc3OUI5NDEwMDk4REIxOEY

01 The basic design of a tower

VIDEO **Citigroup Center — the basic design**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/CitibankSegmentM4MClassroom1.mp4

601 Lexington Avenue opened in 1977: 915 ft tall, 59 floors, about 1.3 million sq ft of office space. Architect Hugh Stubbins, structural engineer William LeMessurier.

A SUSTAINABILITY FEATURE THAT FAILED FOR A GEOMETRIC REASON

The famous 45° sloping roof was designed as a **solar collector** — a flat-plate array to heat water and drive dehumidification, with a gauge in the lobby showing the power coming in. The panels were dropped: at that orientation the collectors would never face the sun squarely enough to pay for themselves.

Worth telling before students optimize anything. A sustainable feature can be defeated by an angle; the geometry has to work, not merely look green.

02 A schematic view of the tower

Model the tower as a **square prism**: base x, height cx, where c is how many base-widths tall it stands.

Surface area (S)	The skin: what you build, clad, insulate, and lose heat through.
Volume (V)	The space enclosed: what you heat, cool, and sell.
Surface-to-volume ratio	S/V — m^2 of skin per m^3 of building. Lower keeps heat in.
Aspect ratio (c)	Height ÷ base width. $c = 1$ is a cube.

> HOW TALL IS CITIGROUP, IN BASE-WIDTHS?
>
> The footprint isn't published directly, so derive it — a good small modeling exercise. 1.3 million sq ft over 59 floors is about 22,000 sq ft per floor; if square, the side is √22,000 ≈ **148 ft**. So $c = 915/148 \approx$ **6.2**.

03 Mathematical analysis

$$V = cx^3 \quad S = (2 + 4c)x^2 \quad S/V = (2 + 4c)/(cx)$$

That expression holds two variables doing completely different jobs, and separating them is the heart of the lesson.

Effect one: size

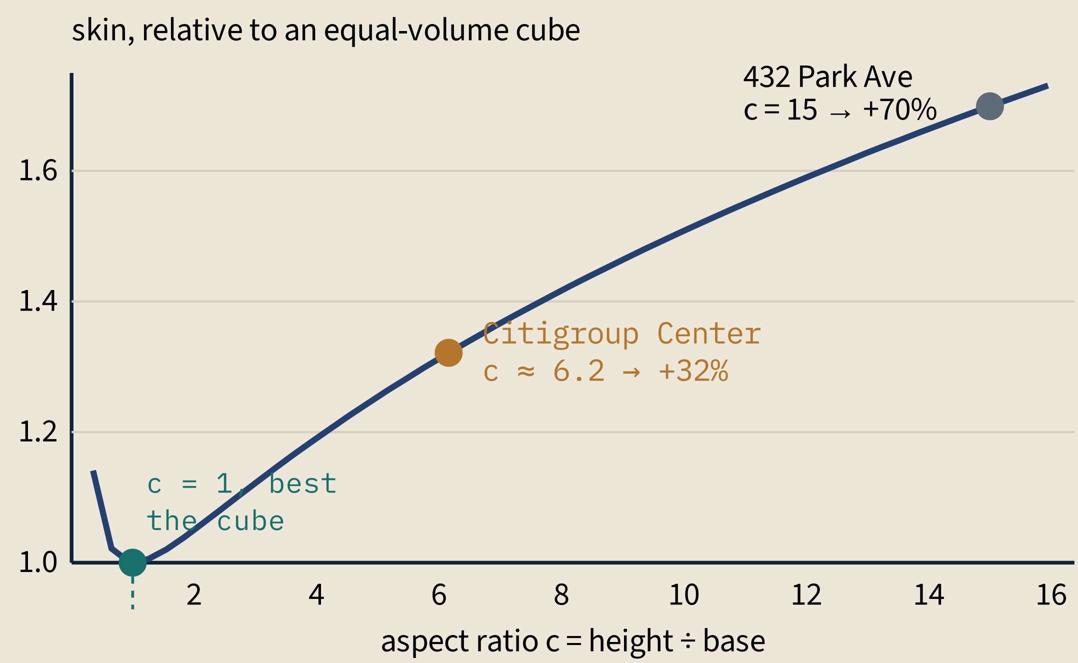
x sits alone in the denominator, so $S/V \propto 1/x$. Double every dimension and the ratio halves, whatever the shape. This is the dominant effect and it is not an architectural choice — it is why a warehouse is cheaper to heat per cubic metre than a shed, and why large animals survive cold better than small ones.

Effect two: shape

This is where the student activity needs care. Varying c with the base fixed makes the ratio fall — but it also makes the building bigger, so effect one is doing the work. It says nothing about shape.

The honest question holds *volume* constant. Then $x = (V/c)^{1/3}$ and:

$$S/V = (2 + 4c) \cdot c^{-2/3} \cdot V^{-1/3}$$



Every shape except the cube costs extra skin — and towers cost a lot.

THE MINIMUM IS THE CUBE, EXACTLY

$$f'(c) = c^{-5/3}(4c - 4)/3$$

Zero at $c = 1$; negative below, positive above — a genuine minimum. Among all square prisms of a given volume, the **cube** has the least surface area. Squash it or stretch it and the ratio climbs either way.

SHAPE	ASPECT RATIO *C*	SKIN VS. AN EQUAL-VOLUME CUBE
Flat slab	0.25	+26%
Low block	0.5	+6%
Cube — the optimum	1	0%
Squat building	2	+5%
Mid-rise	4	?
Citigroup Center	6.2	+32%
432 Park Avenue	15	+70%
Very slender tower	20	+86%

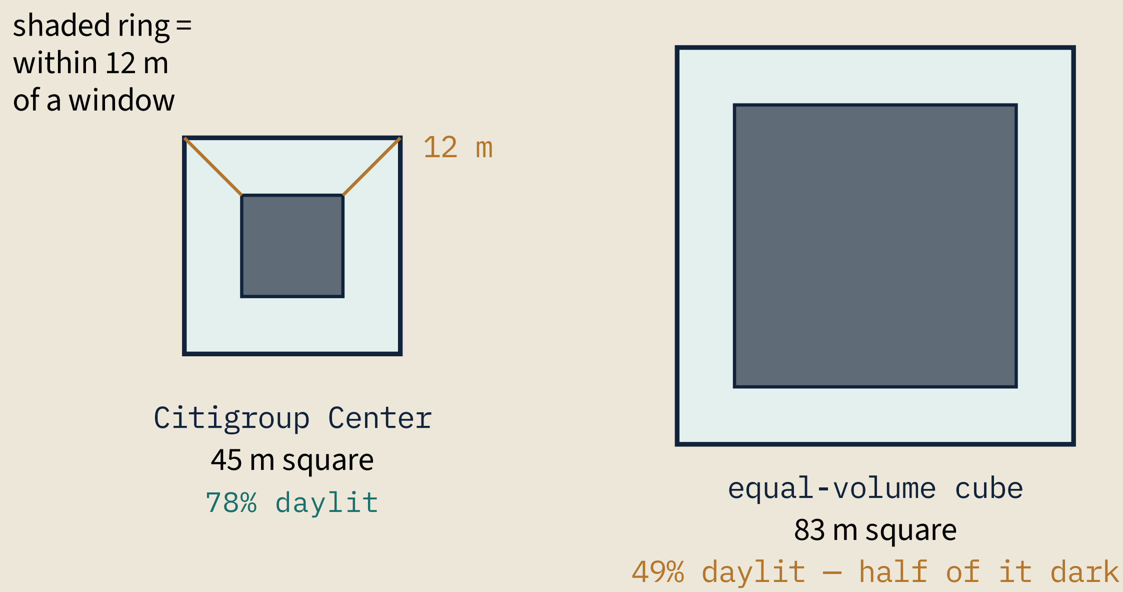
Blank row: +19%. Note the curve is flat near the bottom — $c = 2$ costs only 5%, so a blocky building is nearly as good as a cube. The penalty only bites past about $c = 4$.

CHECK IT AGAINST ANOTHER LESSON

In Architectural Prisms Pt 2, 432 Park Avenue came out at 70% more surface than an equal-volume cube — computed from its actual 93 ft base and 1,396 ft height, a completely different route. Its aspect ratio is 15, and $f(15)/f(1)$ = **1.70** here. Two independent methods, same answer.

04 So why isn't everything a cube?

Because *S/V* is one objective among several, and the shape that minimizes it wrecks another: **daylight**. Office space needs to be within roughly 12 m of a window, so on a square plate of side s the daylit area is $s^2 - (s - 24)^2$.



The cube keeps its heat in beautifully. Half of it has no daylight at all.

TWO OBJECTIVES PULLING OPPOSITE WAYS

Minimizing surface area pushes toward a cube; maximizing daylight pushes toward something thin. They cannot both be satisfied, and the building is the negotiated settlement — Citigroup accepts 32% more skin in exchange for plates that are 78% daylit, on a site that could never have held an 83 m cube anyway.

This is the point to make explicitly: any *single* objective has a clean mathematical answer. A building is what happens when you have four at once. Students who have only ever optimized one function find this genuinely clarifying.

05 Explore it yourself

ACTIVITY **Desmos: the surface-to-volume ratio function**
desmos.com/calculator/pwxthlj1ah
Sliders for c (height relative to base) and b (base length).

TEST THESE, IN THIS ORDER

1. Hold c fixed, increase b. What happens, and why does the curve never reach zero?
2. Hold b fixed, increase c. The ratio falls — but the building also got bigger. Is that a fair comparison?
3. Fix the *volume*. For each c, set $b = (V/c)^{1/3}$ and compare. Where is the minimum?

Question 3 is the one that matters — 1 and 2 measure size, only 3 measures shape. Running them in this order lets students discover the confound themselves rather than being warned about it.

06 Practice, with answers

1. Complete the blank row: a mid-rise with $c = 4$.

ANSWER $f(4) = 18 \times 4^{-2/3} = 7.14$ against $f(1) = 6$, so **+19%**.

2. A cubical building is 30 m on a side. Find S, V, and the ratio; then double every dimension.

ANSWER 30 m: $S = 5{,}400\ \text{m}^2$, $V = 27{,}000\ \text{m}^3$, ratio **0.20**. 60 m: $21{,}600\ \text{m}^2$, $216{,}000\ \text{m}^3$, ratio **0.10**. Surface ×4, volume ×8, ratio **halved** — exactly $S/V \propto 1/x$.

3. A tower has a 40 m square base and is 200 m tall. Find c, S, V, and the ratio.

ANSWER $c = $ **5**; $V = $ **320,000 m³**; $S = 22 \times 1{,}600 = $ **35,200 m²**; ratio **0.11**. Formula check: $22/200 = 0.11$. ✓

4. What cube has that volume, and how much less skin does it have?

ANSWER Side $320{,}000^{1/3} \approx$ **68.4 m**, $S \approx$ **28,100 m²**. The tower has about **25% more** — matching $f(5)/f(1) = 1.25$.

5. How much of that 68.4 m cube's plate is within 12 m of a window?

ANSWER Core 44.4 m → $1{,}971\ \text{m}^2$ dark of $4{,}679\ \text{m}^2$; daylit **58%**. The 40 m tower plate: core 16 m, daylit 1,344 of 1,600 = **84%**.

6. A designer claims taller is always more efficient because the Desmos graph falls as c increases. What's wrong?

ANSWER The graph falls because raising c at fixed base builds a **bigger** building, and bigger always lowers the ratio. It says nothing about shape. Compare at equal volume and the effect reverses — past $c = 1$ the ratio climbs. This is the question to grade carefully; it is the whole lesson.

BACK TO THE OPENING QUESTION

Which shape loses least heat? The **cube**, exactly — $f'(c) = 0$ only at $c = 1$. And no, nobody builds one. Citigroup is six base-widths tall and pays 32% extra skin; 432 Park is fifteen and pays 70%.

They pay because an equal-volume cube would be 83 m across with half its floor in permanent darkness, and because nobody has an 83-metre-square site in midtown Manhattan. Sustainable design is not finding one function's optimum — it is knowing what each optimum costs in the others, to the nearest percent.

Teacher's Guide · Google Earth Voyager Story: The Geometry of Sustainable Architecture, Part 1
Companion to the online lesson. The video, the Desmos activity, and the Voyager story are linked by URL above; the interactive version carries them inline.