Quadrilateral Structures

Leaning towers, centres of gravity, and the two degrees between landmark and catastrophe

Two Madrid skyscrapers lean 15° toward each other, their tops hanging thirty metres out over the road. They have stood there since 1996. The question worth asking is not why they don't fall — it's *how close they are* to falling.

GRADE BAND	TIME	FORMAT
8–10 · Middle School through HS Geometry	**Two to three class periods**	**Video, dynamic geometry construction, stability investigation**

Part 1 covers Stonehenge, Fallingwater, rectangles, and cantilever moments. This part takes up parallelograms and trapezoids and stands alone if needed; §1 reuses Part 1's Stonehenge video.

WHAT STUDENTS WILL BE ABLE TO DO

- Use the properties of parallelograms: opposite sides and angles congruent, adjacent angles supplementary, diagonals bisecting each other.
- Find how far a leaning tower's top is displaced using a tangent ratio.
- Locate the centre of gravity of a leaning building and decide whether it will stand.
- Compute the exact angle at which a tower of given proportions would tip.
- Find the area and midsegment of a trapezoid and apply both to the gateway the towers form.

STANDARDS

7.G.B.6 area of two-dimensional objects · **8.G.A.5** angle relationships · **HSG-CO.C.11** theorems about parallelograms · **HSG-SRT.C.8** right triangles and trigonometric ratios · **HSG-GMD.A.1** and **A.3** volume arguments including Cavalieri's principle · **HSG-MG.A.1** and **A.3** modeling and design problems

START HERE

Open the Voyager story in Chrome and find the Puerta de Europa towers on the Paseo de la Castellana. Have students look from street level, then from directly overhead — where the lean is completely invisible.

Ask: *where did the lean go?* It is a genuinely puzzling observation, and the answer in §4 (every floor is a congruent horizontal square, so the tower is an oblique prism) sets up Cavalieri's principle without anyone having to announce it.

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01 Introduction to quadrilaterals: Stonehenge

VIDEO **Quadrilaterals: Introduction — Stonehenge and post-and-lintel**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/QuadsIntro.mp4
Same clip as Part 1 §1.

The connection the video promises is the one both lessons turn on: **a quadrilateral frame is not rigid.** Stonehenge answered that with weight, pinned joints, and deep footings. In Madrid, Philip Johnson answered it by leaning the frame over on purpose — and then having to prove it would stay there.

02 The geometry of the Puerta de Europa towers

VIDEO **Puerta de Europa, Segment 1 — the towers and centre of gravity**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PuertaDeEuropaSegment1.mp4

Designed by Philip Johnson and John Burgee with engineers Leslie E. Robertson Associates, completed 1996. Each tower is 114 m tall with 26 floors on a 35 m square base, leaning **15° from vertical**.

MEASUREMENT	VALUE
Height	114 m
Square base	35 m on a side
Lean from vertical	15°
Floors	26
Completed	1996

Note: the student lesson says the towers "use rectangular shapes." They don't — that is precisely the point. The elevation has two pairs of parallel sides but no right angles, making it a parallelogram and not a rectangle. Worth correcting out loud.

Parallelogram Both pairs of opposite sides parallel.
Opposite angles Congruent. Here 75° and 75°, 105° and 105°.
Adjacent angles **Supplementary** — 75 + 105 = 180.
Diagonals Bisect each other in any parallelogram; congruent only in a rectangle.
Trapezoid Exactly one pair of parallel sides.
Oblique prism Lateral edges not perpendicular to the bases — a leaning tower.

The 75° and 105° come straight from the lean: 90 − 15 and 90 + 15.

$$\text{displacement} = 114 \times \tan 15^\circ \approx \mathbf{30.5\ m}$$

Published descriptions say the towers reach about 30 m out over the boulevard — a good check that the model matches the building.

03 Constructing a parallelogram with the TI-Nspire

VIDEO **Puerta de Europa, Segment 2 — constructing a parallelogram**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PuertaDeEuropaSegment2.mp4

SHEAR IT AND WATCH WHAT SURVIVES

Build the parallelogram, then drag the top edge sideways keeping it at the same height — a *shear*, exactly what tilting a tower does. Have students track four quantities:

- Slanted sides get **longer**.
- Angles **change**, but adjacent ones stay supplementary.
- Perpendicular height **does not change**.
- Area **does not change** — area = base × perpendicular height.

That last pair is the whole point, and it is the two-dimensional rehearsal for Cavalieri in §4. Students who reach for the slanted side get an answer that is too big, in both dimensions.

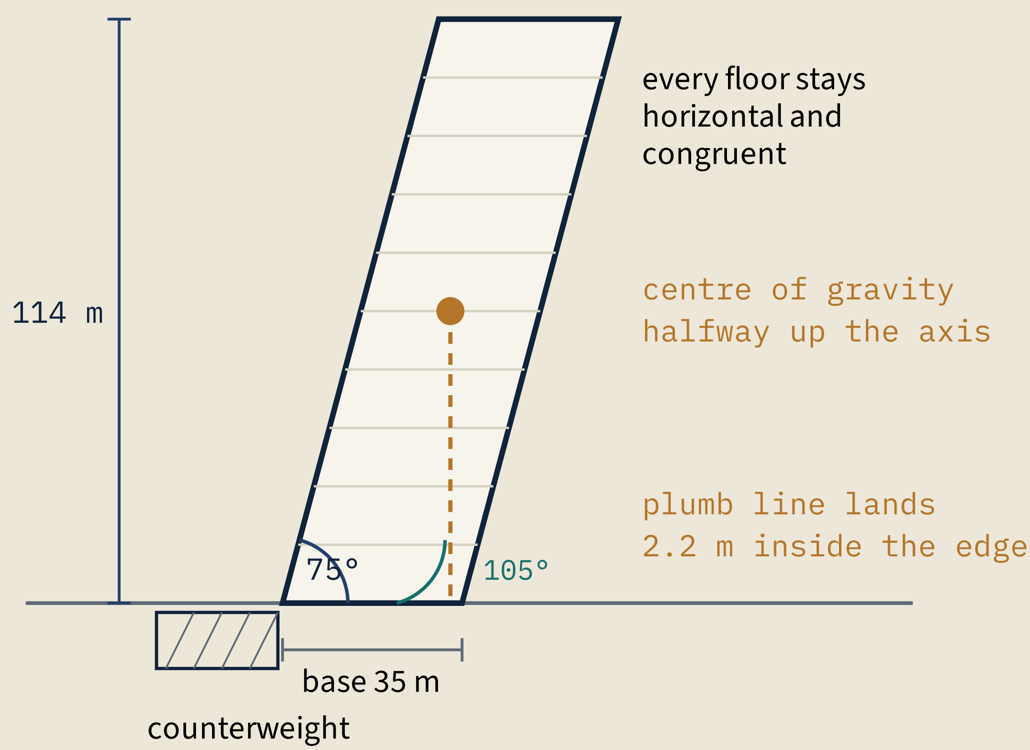
04 Centre of gravity, and how close is too close

VIDEO **Puerta de Europa, Segment 3 — parallelogram structure and trapezoids**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PuertaDeEuropaSegment3.mp4

A building stands as long as a plumb line from its centre of gravity lands inside its base. For a uniform leaning tower the centre of gravity is at the midpoint of the axis — half the height up, half the displacement across:

$$\text{offset} = (114/2) \times \tan 15^\circ \approx \mathbf{15.3\ m}$$

The base reaches 17.5 m either side of centre, so the plumb line lands inside — with **2.2 m** to spare.



Fifteen degrees of lean, and two and a bit metres of margin.

> AT WHAT ANGLE WOULD IT TIP? — THE BEST RESULT IN THE LESSON
>
> Set the offset equal to the half-base. The height cancels:
>
> $$(h/2)\tan\theta = w/2 \;\rightarrow\; \tan\theta = w/h$$
>
> Here $\tan\theta = 35/114 = 0.307$, so $\theta \approx$ **17.1°**. Built at 15°, the margin is about **two degrees** — and that is for an idealized uniform tower with no wind and no uneven loading.
>
> Hence the off-centre concrete core, the steel diagrid, and cables running from the top to a buried 60 × 10 × 10 m counterweight — roughly **14,400 tonnes** of concrete. The geometry says they would just barely stand alone; the engineering makes sure the question never arises. Students find the smallness of the margin genuinely startling, which is the moment to make the point that "stable" and "safe" are not the same word.

Why the plan view hides the lean

Every horizontal slice is the same 35 m square as the base, just shifted — the tower is an **oblique prism**. **Cavalieri's principle** then gives the volume immediately:

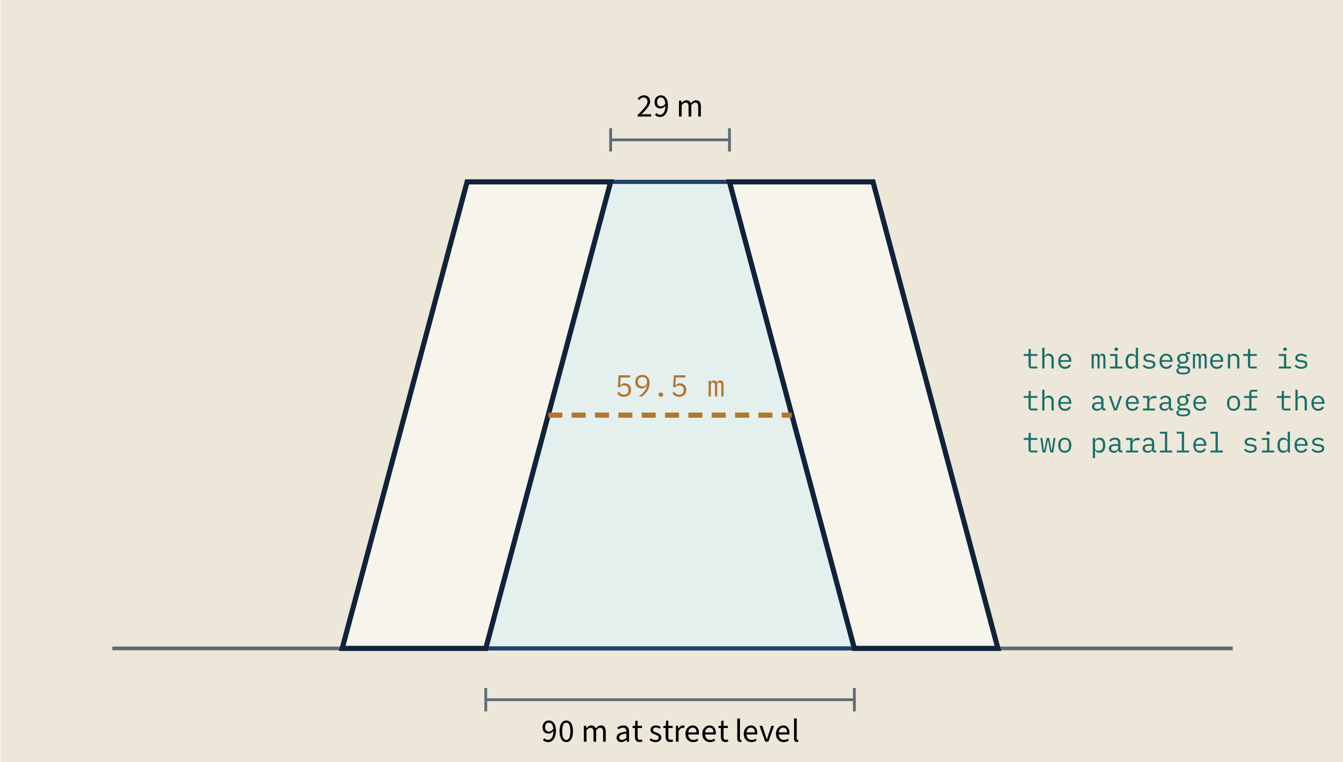
$$V = 35^2 \times 114 \approx 140{,}000\ \text{m}^3$$

Leaning the tower changed nothing about how much building there is. Using the 118 m slant length instead gives 145,000 m^3 — 3.5% too much, the same error as using a parallelogram's slanted side.

The gateway is a trapezoid

Each top swings 30.5 m toward the other, so the opening is wider at the bottom than the top. Assume the towers stand 90 m apart at street level; the gap at roof level is 90 − 61 = 29 m, and the **midsegment** is the average:

$$\text{midsegment} = (90 + 29)/2 \approx \mathbf{59.5\ m}$$



The gate itself is a trapezoid — and its middle width is just the average.

$$A = \tfrac{1}{2}(90 + 29) \times 114 \approx 6{,}800\ \text{m}^2$$

The 90 m street separation is a stated assumption for the arithmetic, not a published figure. The 30.5 m lean per tower is real, so the 61 m narrowing is exact regardless of the gap you choose.

05 Conclusion

VIDEO **Puerta de Europa, Segment 4 — concluding observations**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/PuertaDeEuropaSegment4.mp4

06 Practice, with answers

1. One angle of a parallelogram measures 68°. Find the other three.

ANSWER **68°, 112°, 68°, 112°** — opposite congruent, adjacent supplementary, totalling 360°.

2. A tower 80 m tall leans 12°. How far is its top displaced, and where is its centre of gravity?

ANSWER Top: 80 × tan 12° ≈ **17.0 m**. Centre of gravity is halfway up, so offset 40 × tan 12° ≈ **8.5 m** — half as far.

3. That 80 m tower has a 30 m square base. Is it stable, and at what angle would it tip?

> ANSWER Half-base 15 m against an 8.5 m offset, so **yes**, with 6.5 m of margin — far more comfortable than Madrid's 2.2 m. It tips when $\tan\theta = 30/80 = 0.375$, at **20.6°**.

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4. Two towers of the same height, one 40 m wide and one 20 m wide. Which can lean further?

> ANSWER The **wider** one — $\tan\theta = w/h$, so doubling the width doubles the tangent. Height cancels out entirely: what matters is the *ratio* of width to height, not the size.

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5. Find the volume of a leaning tower with a 25 m square base, 90 m tall, tilted 20°.

> ANSWER **56,250 m³**. The tilt is irrelevant — by Cavalieri every horizontal slice is the same 625 m² square. Using the 95.8 m slant length would give 59,860 m³, about 6% too much.

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6. A trapezoidal opening is 3.2 m wide at the bottom, 1.8 m at the top, 2.5 m tall. Find its midsegment and area.

> ANSWER Midsegment **2.5 m**; area $\frac{1}{2}(3.2 + 1.8) \times 2.5 =$ **6.25 m²**. Note the shortcut: area = midsegment × height, because the midsegment already *is* the average width.

BACK TO THE OPENING QUESTION

Where does the lean go from above? Nowhere — it was never in the plan view. Each floor is a horizontal 35 m square identical to every other, just shifted. That makes the tower an oblique prism, and Cavalieri's principle then hands over the volume without any reference to the angle.

And how close are they to falling? A uniform tower 114 m tall on a 35 m base tips at 17.1°; these lean at 15°. Two degrees of geometric margin is not what holds them up — 14,400 tonnes of buried concrete and a steel cable the height of each building are. But it was the geometry that told the engineers exactly how much help the buildings were going to need.

Teacher's Guide · Google Earth Voyager Story: Quadrilateral Structures, Part 2
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.