# Quadrilateral Structures

Post and lintel, centres of gravity, and why moment grows with the square of the span

Stonehenge's stone beams span about three metres. Fallingwater's concrete terrace reaches fifteen feet with nothing at all under the far end. Both are rectangles. Something changed in 4,500 years — and something didn't.

| GRADE BAND | TIME | FORMAT |
|---|---|---|
| **8–10 · Middle School through HS Geometry** | **Three class periods** | **Video, dynamic geometry construction, load and balance investigation** |

WHAT STUDENTS WILL BE ABLE TO DO

- Find the area of any quadrilateral from its diagonals, and say when that area is largest.
- Show that among all rectangles with a given perimeter, the square encloses the most area.
- Locate the centre of gravity of a rectangle, and explain why it sits where it does.
- Compute the moment on a cantilever and predict what happens when it is lengthened.
- Explain why a stone lintel must be short and a concrete cantilever can be long.

STANDARDS

**7.G.B.6** area of two-dimensional objects · **8.G.A.5** angle relationships · **HSG-CO.C.11** theorems about parallelograms · **HSG-GPE.B.7** perimeters and areas using coordinates · **HSG-C.B.5** arc length · **HSG-MG.A.1**, **A.2** and **A.3** modeling, per-unit reasoning, and design problems

START HERE

Open the Voyager story in Chrome, look at Stonehenge from above, then fly to Fallingwater.

Pose it before any video: *why couldn't Stonehenge's builders make their stone beams longer?* Most students say "stone is heavy" or "they had no cranes." Both are true and neither is the answer. The answer is §7, and it is the same reason Fallingwater nearly fell down.

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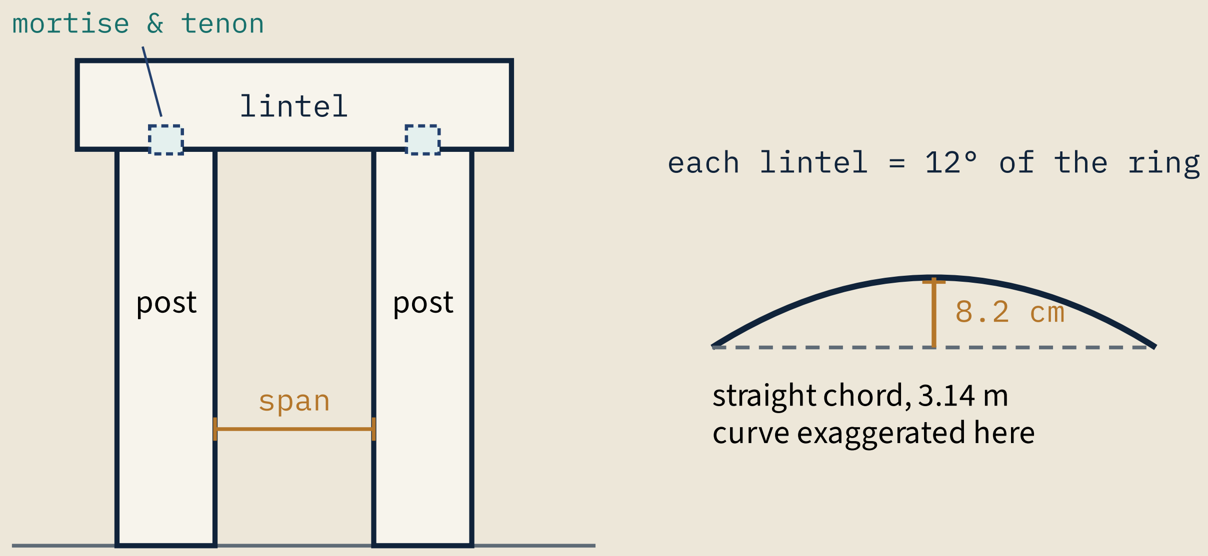
## 01 Introduction to quadrilaterals: Stonehenge

VIDEO **Quadrilaterals: Introduction — Stonehenge and post-and-lintel**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/QuadsIntro.mp4

| | |
|---|---|
| **Post and lintel** | Two uprights carrying a horizontal beam; the opening they frame is a rectangle. |
| **Trilithon** | Greek for "three stones" — two posts and one lintel, standing free. |
| **Span** | The clear distance a beam crosses between supports. |
| **Cantilever** | A beam supported at one end only. |
| **Moment** | Turning effect of a force: force × distance from the pivot. Also called torque. |

Stonehenge's outer sarsen ring was 30 uprights carrying 30 lintels in a continuous circle 30 m across, about 4 m above the ground, each lintel roughly 7 tonnes. They are held by **mortise-and-tenon** joints and locked to each other with **tongue-and-groove** — woodworking joints cut into stone.



Two posts, one beam — and a lintel that had to be carved to a curve.

THE LINTELS ARE NOT STRAIGHT

Thirty lintels share a full turn, so each spans 360°/30 = **12°**. On a 30 m ring that is an arc of π × 30/30 = 3.14 m, matching the published lintel length of about 3.2 m.

The straight chord is 2 × 15 × sin 6° = 3.136 m — only **6 mm** shorter. But the *sagitta*, how far the middle bulges past the chord, is 15(1 − cos 6°) = **8.2 cm**.

Eight centimetres across a face about a metre wide is plainly visible, which is why both faces of every lintel were dressed to a curve. Good moment to make the distinction: over a short arc the *length* barely changes but the *shape* does.

## 02 The geometry of Fallingwater

VIDEO **Fallingwater, Segment 1 — Wright and rectangles**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment1.mp4

Wright designed Fallingwater in 1935 for the Kaufmann family on Bear Run, Pennsylvania. Rather than siting the house to look at the waterfall, he put it *over* the waterfall and threw broad concrete terraces into the air with nothing beneath their far ends.

The main living-room terrace cantilevers about **15 feet**. Write that number on the board; §7 uses it.

## 03 Constructing a quadrilateral with the TI-Nspire

VIDEO **Fallingwater, Segment 2 — constructing a quadrilateral**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment2.mp4

The video lists "formula for the area of any quadrilateral" and "maximum area" without stating either. Both are worth writing down.

### The area of any quadrilateral

$$A = \tfrac{1}{2}\, d_1 d_2 \sin\theta$$

No side lengths needed — just the diagonals and the angle between them. Test it on a rectangle $l \times w$: both diagonals are $\sqrt{(l^2 + w^2)}$ and the sine of their angle is $2lw/(l^2 + w^2)$, so everything cancels to $lw$.

It also gives a maximum: since $\sin\theta \le 1$, fixed diagonals enclose the greatest area when they are **perpendicular**.

### Maximum area for a given perimeter

| LENGTH | WIDTH | PERIMETER | AREA |
|---|---|---|---|
| 1 m | 19 m | 40 m | 19 m² |
| 5 m | 15 m | 40 m | 75 m² |
| 8 m | 12 m | 40 m | 96 m² |
| 10 m | 10 m | 40 m | 100 m² |
| 12 m | 8 m | 40 m | 96 m² |
| 15 m | ? | 40 m | ? |

*Blank row: 5 m and 75 m².* A rectangle of perimeter $P$ has maximum area $P^2/16$ when all sides are equal — the same fact that decided the square base of 432 Park Avenue in the Architectural Prisms series.

## 04 Properties of rectangles

VIDEO **Fallingwater, Segment 3 — properties of rectangles**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment3.mp4

Three side lengths fix a triangle (SSS); four side lengths fix nothing. A rectangular frame can change shape without any member changing length.

SO HOW HAS STONEHENGE STAYED UP FOR 4,500 YEARS?

Not by being rigid — a trilithon is an unbraced quadrilateral frame. It survives by three other means, and naming them turns "quadrilaterals are floppy" from a dead fact into a design question:

- **Weight.** Each upright is about 25 tonnes; friction and mass resist sideways movement.
- **Pinned joints.** Mortise-and-tenon stops the lintel sliding off; tongue-and-groove ties each lintel to its neighbours so the ring acts as one.
- **Deep footings.** The uprights are buried, so each post behaves more like a cantilever fixed in the ground than a pin-jointed strut.

Compare with the Eiffel Tower in *Triangular Structures*, which solves the identical problem by triangulating everything. Two different answers to "stop this frame folding over."

## 05 Centre of gravity

VIDEO **Fallingwater, Segment 4 — centre of gravity and the balcony**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment4.mp4

WHERE IT IS, AND WHY

For a uniform rectangle the centre of gravity is the **intersection of the diagonals** — also the midpoint of each diagonal, and the crossing point of the two lines joining opposite side-midpoints. All three descriptions give the same point.

The proof is symmetry: a rectangle has two lines of symmetry, the balance point must lie on both, and two distinct lines meet in exactly one point. Short enough to do aloud.

The consequence a builder cares about: a uniform slab behaves as though its whole weight hung at that one point. A 15-foot terrace pulls as if everything were 7.5 feet out.

## 06 Finding the centre of gravity with the TI-Nspire

VIDEO **Fallingwater, Segment 5 — centre of gravity on the TI-Nspire**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment5.mp4

DRAG A CORNER AND WATCH

Pull one vertex outward and follow the centre of gravity: it moves, but less than the corner does, because it is an average over the whole shape. Drag the quadrilateral into a long sliver and it slides toward the bulk of the material — which is exactly the judgement a designer makes when placing a heavy stone core against light terraces.

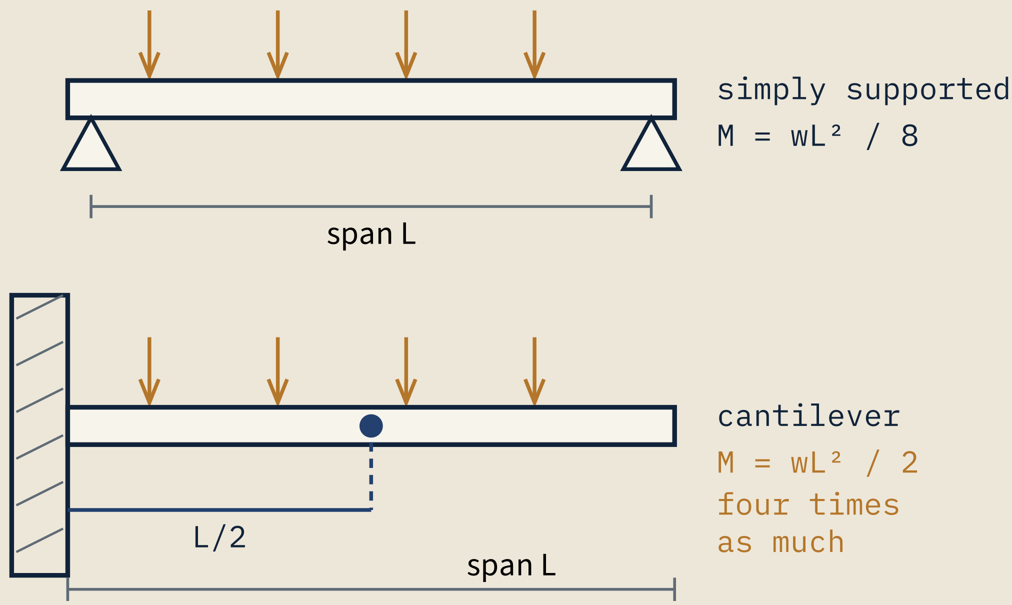
## 07 Cantilevers and torque

VIDEO **Fallingwater, Segment 6 — cantilevers, torque, and reinforced concrete**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment6.mp4

For a uniform slab of length $L$ weighing $w$ per unit length, total weight $wL$ acts at the centre of gravity, $L/2$ from the wall:

$$M = wL \times L/2 = wL^2 / 2$$

The moment depends on the **square** of the length: doubling a cantilever doubles its weight *and* doubles the lever arm, so the root load quadruples. Students reliably predict "twice" — let them, then show them.




Same span, same load. The cantilever's root carries four times the moment.

TWO FORMULAS, ONE EXPONENT — THE CENTRE OF THE LESSON

A beam on two supports carries $wL^2/8$ at midspan. A cantilever of the same span and load carries $wL^2/2$ at its root. Dividing gives exactly **4**.

A cantilever is four times as demanding as a lintel of the same span — the price of removing the far support, and the reason 4,500 years separate Stonehenge from Fallingwater.

But both formulas contain $L^2$. Whether you support a beam at one end or both, the load grows with the *square* of the span. That constraint was identical for Neolithic builders and for Frank Lloyd Wright.

### What it cost Wright

| CANTILEVER LENGTH | RELATIVE MOMENT $(L/15)^2$ |
| --- | --- |
| 10 ft | 0.44 |
| 15 ft — as built | 1.00 |
| 17 ft | 1.28 |
| 20 ft | 1.78 |
| 30 ft | 4.00 |

Wright insisted on light reinforcement; his engineers disagreed, and the contractor quietly added extra steel without telling him. Wright was furious when he found out. Both sides proved too optimistic.

When the formwork came down in 1936 the terrace sagged about 1.75 inches immediately and kept going. By the 1990s the main cantilever had drooped more than **7 inches** over its 15-foot span, cracking parapets and warping window frames, with engineers judging the house at risk of collapse. The 2002 repair — post-tensioned cables threaded through the terraces and anchored into rock — cost over $11 million.

WHERE YOU STAND MATTERS TOO

A 150-pound person on that 15-foot terrace adds a moment that depends entirely on position: **2,250 ft·lb** at the tip, **1,125** halfway out, **450** three feet from the wall. Five times the load from the same person, just for walking to the railing.

This is why cantilevered balconies carry posted occupancy limits — and why the limit is about where people gather as much as how many there are.

## 08 Conclusion

VIDEO **Fallingwater, Segment 7 — why quadrilaterals matter to architects**
s3.amazonaws.com/Media4MathPlus/Videos/GoogleVoyagerVideos/FWSegment7.mp4

## 09 Practice, with answers

**1. Complete the blank row: perimeter 40 m, one side 15 m.**

ANSWER Half the perimeter is 20 m, so the other side is **5 m**, area **75 m²** — same as the 5 × 15 row, since a rectangle turned on its side is the same rectangle. The 10 × 10 square still wins at 100 m².

**2. A quadrilateral has diagonals 12 cm and 9 cm meeting at 40°. Find its area, and the largest area those diagonals could give.**

ANSWER ½ × 12 × 9 × sin 40° ≈ **34.7 cm²**. Maximum at 90°: **54 cm²** — over 55% more from the same two diagonals, just by changing the angle.

**3. A slab cantilevers 12 ft and weighs 100 lb per foot of length. Find the total weight, where it acts, and the moment at the wall.**

ANSWER Weight **1,200 lb** acting **6 ft** out; moment 1,200 × 6 = **7,200 ft·lb**. Check: $wL^2/2$ = 100 × 144/2 = 7,200. ✓

**4. The same slab is redesigned at 18 ft. By what factor does the moment increase?**

ANSWER $(18/12)^2$ = **2.25 times**. Students who answer 1.5 have forgotten that the weight grows as well as the lever arm — the most common error in this lesson.

**5. A lintel and a cantilever both span 3 m with the same load per metre. Which carries the greater moment, and by how much?**

ANSWER The **cantilever**, by exactly **4**. Removing one support does not make things somewhat harder; it quadruples the demand at the one that remains.

**6. What if Stonehenge's 30 m ring had been built with 20 longer lintels instead of 30?**

ANSWER Each would span 18° — an arc of π × 30/20 = **4.71 m**. Since moment goes as $L^2$, each would carry $(4.71/3.14)^2$ = **2.25 times** the bending. In stone, weak in tension, that is a very good reason to prefer thirty short lintels.

BACK TO THE OPENING QUESTION

Why couldn't Stonehenge's builders make their beams longer? Because bending grows with the **square** of the span and stone is weak in tension — a stone beam cracks on its underside long before it is crushed. Thirty short lintels was arithmetic, not lack of ambition.

What changed in 4,500 years is the material: steel takes tension, so reinforced concrete let Wright throw a slab fifteen feet into the air with nothing under it. What didn't change is the $L^2$. Wright pushed against the same exponent the Neolithic builders did, and it pushed back — seven inches of sag and an $11 million repair.

Teacher's Guide · Google Earth Voyager Story: Quadrilateral Structures
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.