The Geometry of Castles

Why sight distance grows as the square root of height — and what that bought a castle

A watchtower buys one thing: the time between seeing an army and meeting it. How much time depends on the tower's height — but not in the way anyone guesses, because the Earth curves away underneath.

GRADE BAND	TIME	FORMAT
9–11 · High School Geometry	**Two class periods**	**Video, dynamic geometry construction, distance calculation**

WHAT STUDENTS WILL BE ABLE TO DO

- Use the fact that a tangent meets a circle perpendicular to the radius at the point of tangency.
- Derive the line-of-sight distance from a height above a sphere using the Pythagorean Theorem.
- Show that sight distance grows with the *square root* of height, and say what that means for design.
- Convert a sighting distance into hours of warning.
- Apply the same geometry to a satellite, and explain why the tower shortcut breaks down there.

STANDARDS

8.G.B.7 applying the Pythagorean Theorem · **HSG-C.A.2** relationships among radii, chords, and tangents, including the right angle at the point of tangency · **HSG-SRT.C.8** right triangles and trigonometric ratios · **HSF-IF.C.7b** graphing square root functions · **HSA-CED.A.2** creating equations in two variables · **HSG-MG.A.1** and **A.3** modeling and design problems

START HERE

Open the Voyager story in Chrome and view a castle from the height of its tallest tower. Then get a written prediction before anything else:

If you double the height of a watchtower, how much further can the sentries see? Classes split between "twice" and "four times." Both are wrong — it is √2, about 41% — and collecting the wrong guesses first is what makes the square root memorable.

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earth.google.com/web/data=Mj8KPQo7CiExelhQOGM3bjhEanR6NGFFakctM3lyWWg2OG90VlQ0Rkw
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BEFORE YOU TEACH THIS — TWO MISSING VIDEOS

The student lesson says "watch this video about Himeji Castle" (§1) and "watch the video to learn how to calculate the line of sight distance" (§2). **Neither video is present in the markup** — both places hold an empty paragraph where an embed should be.

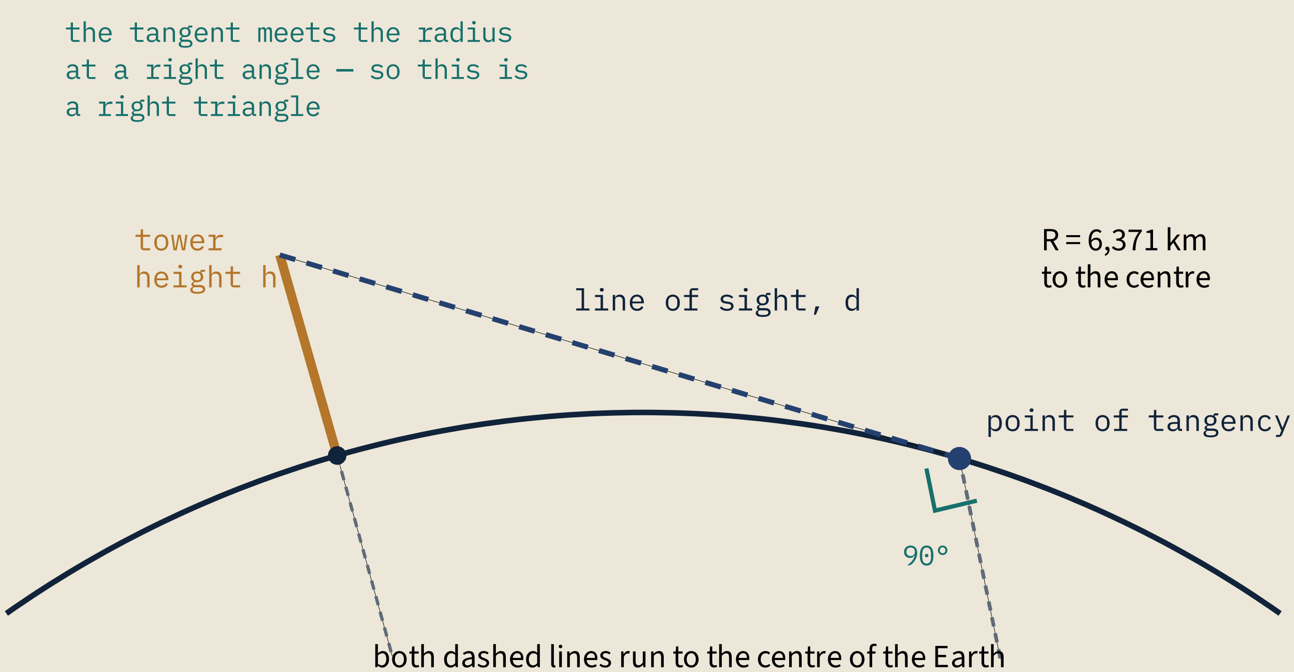
The enhanced lesson carries the full derivation in text and diagrams, so it stands alone without them. But if those clips exist, they should be restored; and if they do not, the §1 and §2 sentences referring to them need rewording on the live page.

01 Angles and line of sight

Many castles had a circular watchtower giving a wide view. Because the Earth curves away, a sentry's line of sight reaches only as far as the point where it grazes the surface — the **point of tangency**.

Tangent line	A line touching a circle at exactly one point.
Point of tangency	That single point of contact. For a sentry, the horizon.
Tangent–radius theorem	A tangent is **perpendicular** to the radius at the point of tangency — the fact that makes the whole calculation possible, and the one the student lesson never states.
Line of sight	The straight distance from observer to horizon.

02 Calculating the line-of-sight distance



Legs d and R, hypotenuse $R + h$. Everything follows from that.

$$d^2 + R^2 = (R + h)^2 \;\rightarrow\; d = \sqrt{(2Rh + h^2)}$$

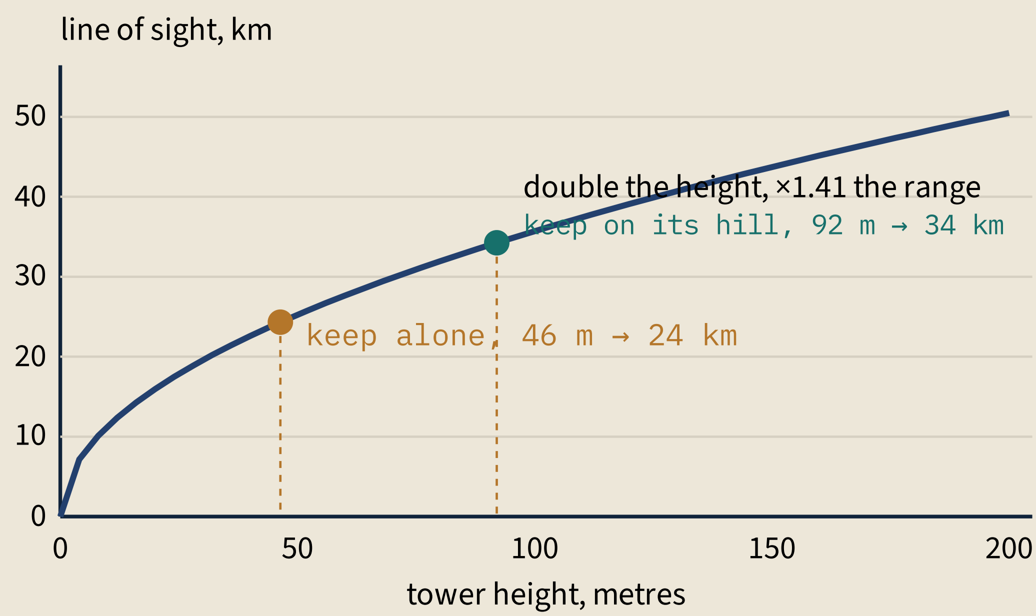
For a castle h^2 is about a millionth of $2Rh$. Drop it, put h in metres and d in kilometres:

$$d \approx 3.57\,\sqrt{h}$$

THE SQUARE ROOT CHANGES EVERYTHING

Doubling the height does **not** double the sight distance — it multiplies it by $\sqrt{2} \approx 1.41$. To see twice as far you need **four times** the height; three times as far, nine times the height.

This is a real constraint on castle design, and it explains why builders reached for hilltops. When every extra metre buys less than the one before, free height from terrain is the best kind there is.



Steep at first, then stubbornly flat. The first ten metres are worth more than the next hundred.

03 Himeji Castle, in numbers

Himeji is the finest surviving Japanese castle — 83 buildings, UNESCO listed, built up between 1333 and 1609. The main keep is **46.4 m** tall on Himeyama hill, itself **45.6 m** high: **92 m** above sea level. Those figures are almost exactly a factor of two apart, which makes Himeji a natural test of the rule.

VANTAGE POINT	HEIGHT	LINE OF SIGHT	WARNING TIME*
A rider on the plain	2 m	5.1 km	1.3 h
A curtain wall	10 m	11.3 km	2.8 h
The keep alone	46.4 m	24.3 km	6.1 h
The keep on its hill	92 m	34.2 km	8.6 h
A hypothetical tower	150 m	?	?
Twice the hill height again	200 m	50.5 km	12.6 h

Blank row: 43.7 km, 10.9 hours. * assumes an army marching at 4 km/h.

Check the highlighted rows: 92/46.4 = 1.98 and 34.2/24.3 = **1.41**. Doubling the height multiplied the range by √2 exactly as predicted — and the hill is doing as much work as the entire six-storey keep. This is the moment to return to the opening prediction.

WHAT THE TOWER ACTUALLY BOUGHT

From a 10 m wall an army appears 11.3 km off — under three hours' warning. From the keep it appears 34.2 km off, giving **8.6 hours**. That difference of roughly **six hours** is the whole point of the tower: enough to close the gates, call in the outlying farmers, light signal fires, and get archers onto the walls.

The geometry converts height into time, and time is what wins a siege. Students who have only met the Pythagorean Theorem as an abstract exercise tend to find this the most persuasive application in the course.

A REFINEMENT: YOU SEE THE TOPS OF THINGS FIRST

$$d_{\text{total}} \approx 3.57(\sqrt{h_1} + \sqrt{h_2})$$

An army is not flat on the ground. With banners 3 m up: 34.2 + 6.2 = **40.4 km**, or 10.1 hours. This is why ships see each other's masts before their hulls, and why signal fires were built high at *both* ends of a chain.

04 From watchtowers to satellites

WHERE THE SHORTCUT BREAKS — WORTH THE DETOUR

The rule $d \approx 3.57\sqrt{h}$ came from discarding h^2. Safe for a 92 m tower, where h^2 is a millionth of $2Rh$. *Not* safe for a satellite: for GPS at 20,200 km, $2Rh \approx 257$ million while $h^2 \approx 408$ million — the discarded term is now the **larger** one.

Worth stating as a general lesson: an approximation is valid only where its dropped term is small, and nothing in the algebra warns you when you have left that range. Students routinely carry shortcuts outside their domain; this is a vivid, checkable example.

VANTAGE POINT	ALTITUDE	HORIZON DISTANCE	SHARE OF EARTH IN VIEW
Himeji keep	92 m	34 km	0.00001%
International Space Station	420 km	2,351 km	3.1%
GPS satellite	20,200 km	25,796 km	38%

The visible share comes from the same triangle: $\cos\theta = R/(R+h)$, and the cap covers $(1-\cos\theta)/2$ of the sphere. Three GPS satellites spaced around the planet see essentially all of it — which is how the system is arranged.

05 Practice, with answers

1. Complete the blank row: a 150 m tower.

ANSWER $3.57\sqrt{150} \approx$ **43.7 km**, or **10.9 hours**. Going from 92 m to 150 m — 58 extra metres of tower — bought only 2.4 more hours.

2. A sentry can see 20 km. How high is the tower?

ANSWER $h = (20/3.57)^2 = 5.60^2 \approx$ **31 m**. Squaring undoes the square root — the step students most often skip.

3. How tall must a tower be to see twice as far as a 25 m one?

ANSWER **100 m** — four times the height. 25 m sees 17.8 km; 100 m sees 35.7 km. To multiply range by k, multiply height by k^2.

4. Use the full formula for a 92 m tower and compare with the shortcut.

ANSWER In kilometres, $h = 0.092$: $2Rh = 1{,}172.264\ \text{km}^2$, $h^2 = 0.00846\ \text{km}^2$, so $d = \sqrt{1{,}172.272} \approx$ **34.24 km** — matching the shortcut to four figures. Good unit-discipline practice; the mixed metres-and-kilometres is where errors appear.

5. A geostationary satellite orbits at 35,786 km. Find its horizon distance and the fraction of Earth it sees.

ANSWER $d = \sqrt{(455{,}985{,}212 + 1{,}280{,}637{,}796)} \approx$ **41,673 km** — note h^2 is nearly three times $2Rh$ here. $\cos\theta = 6{,}371/42{,}157 = 0.1511$, $\theta = 81.3°$, cap $\approx$ **42%**. Three of them cover the inhabited world, which is why weather and television satellites sit there.

6. Two castles have keeps 40 m and 90 m tall, 30 km apart. Can each see the other?

ANSWER $3.57(\sqrt{40} + \sqrt{90}) = 3.57(6.32 + 9.49) \approx$ **56.4 km**, so **yes**, comfortably. This is how chains of signal towers worked: each only had to reach its neighbour, and both ends being high is what made the spacing possible.

BACK TO THE OPENING QUESTION

Double the height and how much further can you see? Neither twice nor four times — **1.41 times**. Himeji proves it: 46.4 m gives 24.3 km, 92 m gives 34.2 km, a ratio of exactly $\sqrt{2}$.

So were castles tall for symbolic reasons? Partly. But Himeji's keep converted its height into roughly six extra hours of warning over a curtain wall, and six hours is the difference between a defended castle and an overrun one. The square root is also why so many castles sit on hills: nature supplies height for free, and when every additional metre buys less than the last, free height is the best kind there is.

Teacher's Guide · Google Earth Voyager Story: The Geometry of Castles
Companion to the online lesson. The Voyager story is linked by URL above; see the note on p.1 regarding the two missing video embeds.