# Circular Structures

Kivas, foci, and the arcs that stand in for the Colosseum's ellipse

A Roman surveyor had rope and stakes. With those you can strike a perfect circle all day long — but you cannot draw an ellipse. So how did they lay out the Colosseum?

| GRADE BAND | TIME | FORMAT |
|---|---|---|
| 9–10 · High School Geometry | Three class periods | Video, construction, calculation, physical model |

WHAT STUDENTS WILL BE ABLE TO DO

- Name and use the parts of a circle: center, radius, diameter, chord, arc, central angle.
- State the locus definitions of a circle and an ellipse, and explain what the foci do.
- Find the foci, eccentricity, and area of an ellipse from its axes.
- Build a four-centered oval from circular arcs and measure how closely it matches a true ellipse.
- Explain why tangency — not just closeness — is what makes arcs join smoothly.

STANDARDS

**7.G.B.4** area and circumference of a circle *(prior knowledge)* · **HSG-C.A.1** all circles are similar · **HSG-C.A.2** relationships among central angles, radii, and chords · **HSG-C.B.5** arc length and sectors · **HSG-GPE.A.3(+)** equations of ellipses from the foci · **HSG-CO.D.12** formal geometric constructions · **HSG-MG.A.1** and **A.3** modeling with geometry and design problems

START HERE

Open the Voyager story in Chrome and look down on both sites: the kivas at Pueblo Bonito and the Colosseum in Rome.

Then do this at the board with a pencil and a loop of string. Pin one end — a circle appears instantly. Now ask a student to draw an ellipse with the same tools. The failure is the hook: *what do you need that one pin and string won't give you?* §3 answers it.

earth.google.com/web/data=MkEKPwo9CiExMmVvUWNPbmJoYndkVWdoTDRQTjF6amZkYmhreGdPTDESFgoUMDlFN0FFMzExRjJBRTIxNTQ3RjkgAQ

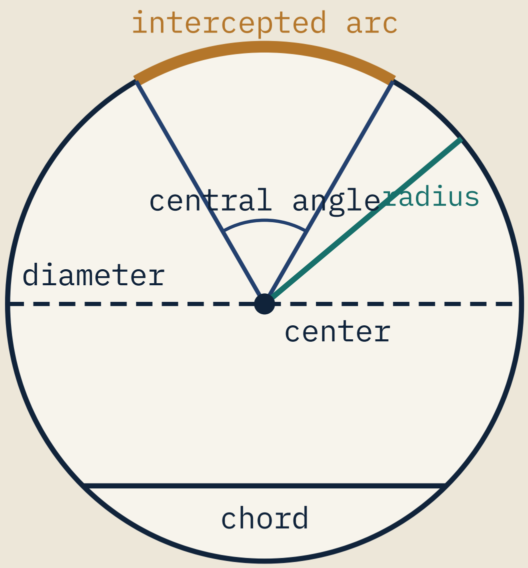
## 01 Introduction to circles: Pueblo Bonito

VIDEO **Pueblo Bonito — circles and the sky**
content.jwplatform.com/players/D8OCCi9S-Y4Ue9e9A.html

Pueblo Bonito was built in stages between about 850 and 1150 CE by Ancestral Puebloan people. It is a D-shaped great house of 600 to 800 rooms, holding three great kivas and more than thirty smaller ones. A wall runs north to south across the central plaza, aligned so precisely to true north that at solar noon it casts almost no shadow.

The video introduces five terms but does not define them. They are collected below — worth putting on the board before playing the clip, since the rest of the lesson leans on them.

| | |
|---|---|
| **Center** | The fixed point every point on the circle is equidistant from. |
| **Radius** | A segment from the center to the circle; also its length. |
| **Diameter** | A chord through the center — twice the radius, and the longest chord a circle has. |
| **Chord** | A segment joining any two points on the circle. |
| **Arc** | A piece of the circle itself — part of the curve, not of the interior. |
| **Central angle** | An angle with its vertex at the center. The arc it opens onto is the **intercepted arc**. |




A kiva is a circle. Every point on the wall is the same distance from the center, so no direction of the sky is privileged.

Every term the video introduces, on one circle.

WHY OBSERVATORIES ARE ROUND

A circle is the set of all points a fixed distance from a center — a **locus**. A rectangular room has corners and long walls, so sightlines toward different horizons meet the wall at different distances and angles. In a circle they all meet it the same way. The shape removes the building's own bias from the observation.

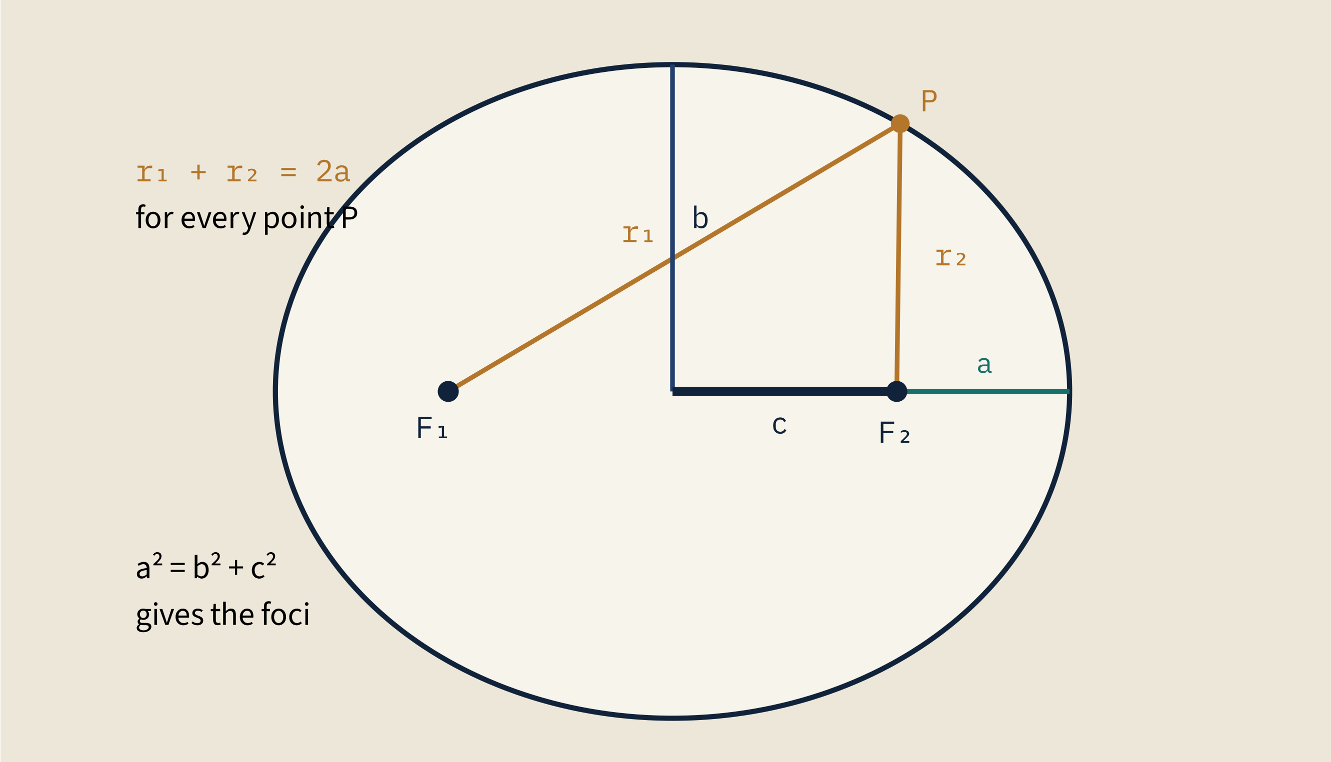
This is also the cleanest available answer to "why do we care about the locus definition?" — it is the property doing the work.

## 02 The geometry of the Roman Colosseum

VIDEO **The Roman Colosseum and circular arcs**
content.jwplatform.com/players/DQ4Y3E8A-Y4Ue9e9A.html

| MEASUREMENT | VALUE | SEMI-AXES |
|---|---|---|
| Outer footprint | 189 m × 156 m | $a$ = 94.5, $b$ = 78 |
| Arena floor | 87 m × 55 m | $a$ = 43.5, $b$ = 27.5 |
| Façade height | ≈ 48 m | — |
| Ground-level arches | 80 | — |

**Locus of points** The set of all points satisfying a condition. Circles and ellipses are both defined this way.
**Circle** The locus of points at a fixed distance from *one* point.
**Ellipse** The locus of points whose distances to *two* fixed points add to a constant.
**Focus (foci)** Either of those two points. Slide them together and the ellipse becomes a circle.
**Radius (radii)** In an ellipse, the segments from the foci to a point are the focal radii.
**Circular arc** A connected piece of a circle — every point the same distance from that circle's center.




Two pins and a loop of string draw an ellipse. One pin draws a circle.

$$c = \sqrt{(94.5^2 - 78^2)} = \sqrt{2{,}846} \approx 53.4 \text{ m}$$

The foci sit 53.4 m either side of center — about 107 m apart, roughly a football field. Eccentricity $c/a \approx 0.56$: a decidedly stretched ellipse, not a nearly circular one.

## 03 Arcs, not an ellipse

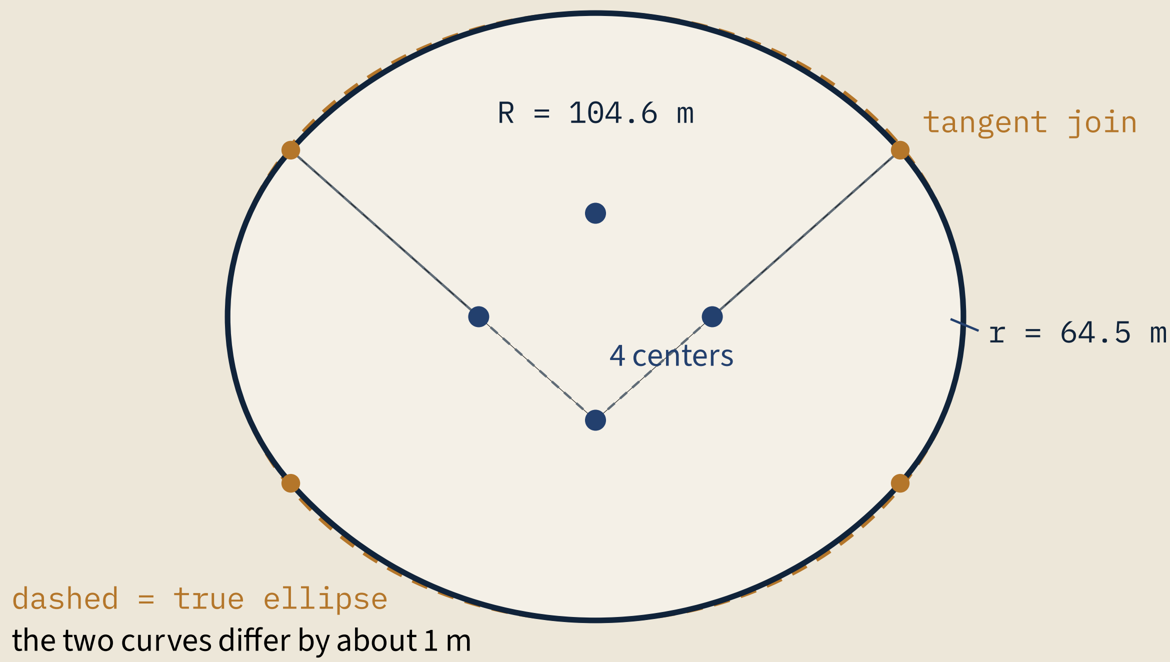
This section is new material and the heart of the lesson. The original states that the Colosseum was built from circular arcs but never shows what that means or why it matters.

A surveyor laying out a 189 m building had rope, stakes, and labor. Pin a rope and walk it around: a circle of any radius. The two-pin string trick works on paper, but at building scale a rope that must stay taut while sliding around two stakes stretches, snags, and drifts. So Roman builders used a chain of circular arcs — a **four-centered oval**: two tight arcs at the ends, two broad arcs along the flanks.

### Building one

- Each arc must reach its vertex: the end arc through $(a, 0)$, the flank arc through $(0, b)$.
- Where two arcs meet they must be **tangent** — and two circles are tangent exactly when the join point lies on the line through both centers.

That second condition is the whole trick, and it is why the §4 construction spends its time on tangents. Arcs that merely touch leave a visible kink; tangent arcs share a direction at the join.




Four arcs, four centers. The dashed ellipse is underneath — if you can find it.

With centers at (±30, 0) and (0, ±26.6), the end arcs have radius **64.5 m** and the flank arcs **104.6 m**. The largest gap between that curve and the true ellipse is about **1 metre** — roughly 1% of the semi-major axis, on a building 189 m long.

WHY BOTHER? COUNT THE TEMPLATES.

An ellipse curves by a different amount at every point; a circular arc curves the same amount everywhere. That decides how many distinct stones must be cut.

Eighty arches ring the ground level; a quarter of the building holds 20. On a true ellipse each of those 20 sits on differently-curved ground — 20 distinct templates. On a four-centered oval there are only two radii in the whole building — **2 templates**.

Same idea as the equilateral truss in *Triangular Structures*: one cut, one jig, repeated. Roman engineering was mass production, and the geometry was chosen to make it possible. If students have done that lesson, name the connection.

AN HONEST COMPLICATION — WORTH TEACHING, NOT HIDING

Scholars have argued about this for a century. Cozzo proposed a four-centered oval; Trevisan's precise modern survey fits an eight-centered oval, or a true ellipse, slightly better.

The reason the argument resists settlement is mathematical: once the axes are fixed, all three curves agree to within a metre or two, and the building has been losing stone for 1,950 years. The differences between the theories are smaller than the damage. **When two models agree to inside your measurement error, no amount of measuring will separate them.** That is a genuinely important idea and students rarely meet it in a geometry class.

## 04 Constructing a model with the TI-Nspire

VIDEO **Constructing the Colosseum framework, keystroke by keystroke**
content.jwplatform.com/players/2eNnJYlP-Y4Ue9e9A.html

The construction uses the **Cartesian coordinate plane**, **line segments**, **midpoint**, **intersection point**, **tangent**, and **slope**. The target sketch is at s3.amazonaws.com/Media4MathPlus/CustomPages/CustomPageGraphics/ArcSegmentsColliseum-01.png

WATCH FOR THE TANGENT STEP

Midpoint, intersection, and slope all serve one goal: arcs that meet tangentially, sharing a slope so the curve has no corner.

Have students test it — drag a center slightly off the line through the join point and a kink appears. Tangency is not a finishing touch; it is the condition that makes a chain of arcs read as one smooth curve.

## 05 Building a 3D model

VIDEO **Building the Colosseum from blocks**
content.jwplatform.com/players/qjsYAHWo-Y4Ue9e9A.html

HANDOUT **Blueprint image**
s3.amazonaws.com/Media4MathPlus/CustomPages/CustomPageGraphics/ArcSegmentsColliseum2-01.jpg

SCALE THE MODEL BEFORE BUILDING

Decide the scale first and write it down. A 60 cm long axis against a real 189 m gives 0.60/189 = **1 : 315**. Then the short axis *must* be 156/315 = 49.5 cm, not whatever looks right — and the two arc radii scale identically: 64.5 m → 20.5 cm, 104.6 m → 33.2 cm. Two compass settings build the entire footprint, which is the §3 argument made physical.

## 06 Practice, with answers

**1. A great kiva is 64 ft across. Find its circumference and floor area.**

ANSWER Radius 32 ft. Circumference = π × 64 ≈ **201 ft**; area = π × $32^2$ ≈ **3,217 ft²**. A square room with the same 201 ft perimeter encloses only about 2,525 ft² — the circle wins, and that is a general result.

**2. Find the foci and eccentricity of the Colosseum's outer ellipse.**

ANSWER $c = \sqrt{(94.5^2 - 78^2)}$ ≈ **53.4 m** either side of center on the long axis. Eccentricity 53.4/94.5 ≈ **0.56**.

**3. What fraction of the Colosseum's footprint is the arena floor?**

ANSWER Outer π × 94.5 × 78 ≈ 23,160 m²; arena π × 43.5 × 27.5 ≈ 3,760 m². The arena is about **16%** — the other 84% is seating, stairs, and structure.

**4. Estimate the perimeter of the outer ellipse and compare with the commonly quoted 527 m.**

ANSWER Ramanujan's approximation gives **≈ 543 m**, about 3% above the quoted figure. Worth asking why rather than assuming an error: published perimeters may follow a slightly different curve, or come from a different survey. Two numbers disagreeing by 3% are usually measuring two slightly different things.

**5. Verify the tangency condition for the four-centered oval (end arcs r = 64.5 m at (±30, 0); flank arcs R = 104.6 m at (0, ∓26.6)).**

ANSWER Internal tangency means the distance between centers equals the difference of radii. Distance from (30, 0) to (0, −26.6) = √(900 + 708) ≈ **40.1 m**; 104.6 − 64.5 = **40.1 m**. They match, so the join is invisible.

**6. Why can't a single circular arc, of any radius, be part of an ellipse?**

ANSWER Curvature along an arc is **constant**; an ellipse's curvature **changes continuously**, tightest at the ends of the long axis and gentlest at the ends of the short axis. Two curves agreeing along an arc would need equal curvature there, so no arc of an ellipse is ever circular. The oval can match to within a metre but never coincide.

BACK TO THE OPENING QUESTION

What do you need that one pin and string won't give you? A second pin — and at building scale that is exactly what the Romans had no good way to use. So they replaced the ellipse with a chain of circular arcs any surveyor could strike with a rope, chosen so that the substitute sits within a metre of the real thing across a 189-metre span.

It bought them something an ellipse never could: a building made of two kinds of stone block instead of twenty. Nineteen and a half centuries later, we still can't measure the difference well enough to be sure which curve they used.

Teacher's Guide · Google Earth Voyager Story: Circular Structures, Part 1
Companion to the online lesson. Videos, the blueprint, and the Voyager story are linked by URL above; the interactive version carries them inline.