Circular Structures

Inscribed angles, radians, and calculating where the Pantheon's sunbeam lands

The Pantheon has one window: a hole in the roof, 8.2 metres across. Once a year at noon the beam through it lands squarely on the doorway. Students can work out which day that is with a right triangle.

GRADE BAND	TIME	FORMAT
9–10 · High School Geometry	**Three class periods**	**Video, construction, arc-length calculation, solar modeling**

Part 1 of this series covers Pueblo Bonito and the Colosseum, and defines the circle vocabulary §1 here reviews. The two parts share a Voyager story and the §1 video.

WHAT STUDENTS WILL BE ABLE TO DO

- State and use the Inscribed Angle Theorem: an inscribed angle is half the central angle on the same arc.
- Convert between degrees and radians, and find arc length with $s = r\theta$.
- Turn a chord into a central angle, and a central angle into an arc length.
- Model a shaft of sunlight as a line and predict exactly where it strikes the inside of a dome.
- Explain why every landing spot in the Pantheon corresponds to *two* dates in the year.

STANDARDS

HSG-C.A.2 relationships among inscribed angles, central angles, radii, and chords · **HSG-C.B.5** arc length, radian measure, and sectors · **HSG-CO.D.12** formal geometric constructions · **HSG-SRT.C.8** right triangles and trigonometric ratios · **HSF-TF.A.1** radian measure · **HSG-MG.A.1** modeling real objects with geometric shapes

START HERE

Open the Voyager story in Chrome and find the Pantheon from above. The dark circle in the roof is the oculus — the building's only window.

Then pose it: the sun moves, the hole doesn't, so the patch of light must move, and where it lands depends on the date. *Is there a day when it lands on the front door — and could the Romans have chosen it?* §6 has students calculate the answer rather than look it up.

earth.google.com/web/data=MkEKPwo9CiExMmVvUWNPbmJoYndkVWdoTDRQTjF6amZkYmhreGdPTDESFgoUMDlFN0FFMzExRjJBRTIxNTQ3RjkgAQ

01 Review: circles at Pueblo Bonito

VIDEO **Pueblo Bonito — circles and the sky**
`content.jwplatform.com/players/D8OCCi9S-Y4Ue9e9A.html`
Same clip as Part 1 §1. A two-minute refresher if the class has seen it; longer if not, since everything below rests on "central angle" and "intercepted arc."

Center	The fixed point every point on the circle is equidistant from.
Radius	A segment from the center to the circle; also its length.
Diameter	A chord through the center — twice the radius.
Chord	A segment joining any two points on the circle.
Central angle	An angle with its vertex at the center.
Intercepted arc	The piece of the circle an angle opens onto.

Note: the student lesson's §1 list omits "central angle," though §2 relies on it. It is included above.

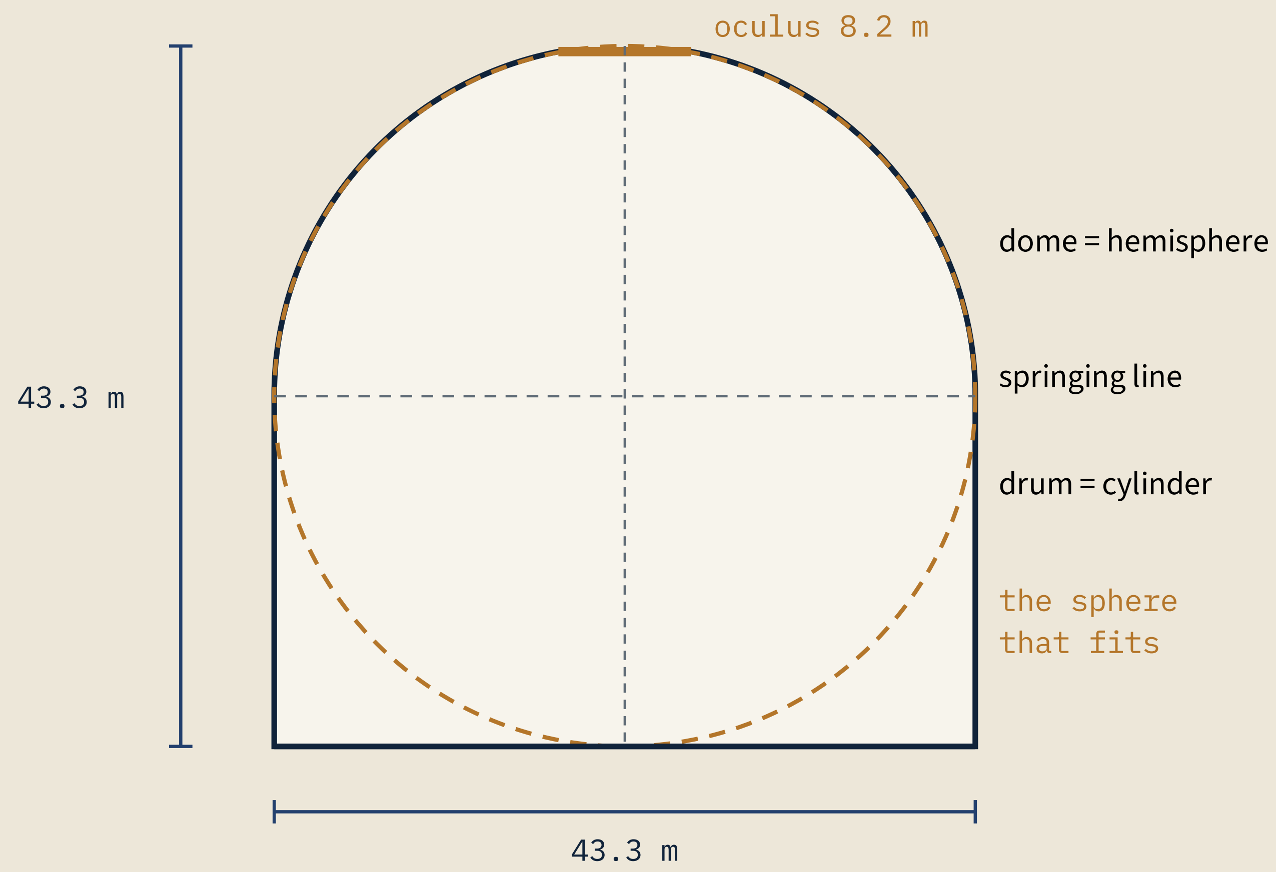
02 The geometry of the Roman Pantheon

VIDEO **The Roman Pantheon and the properties of circles**
`content.jwplatform.com/players/nRvGp45D-Y4Ue9e9A.html`

The Pantheon standing today was built under Hadrian around 118–128 CE. Nineteen centuries later its dome is still the largest unreinforced concrete dome on Earth.

MEASUREMENT	VALUE
Interior diameter	43.3 m
Floor to oculus	43.3 m — the same number
Oculus diameter	8.2 m
Coffers	5 rings of 28
Built	c. 118–128 CE

Those first two rows are the whole design. Width equals height, so a **sphere 43.3 m across fits exactly inside**, touching the floor at one point and the oculus at the other. The building is a cylinder with a hemisphere on top, and the hemisphere's radius equals the cylinder's height.



Width equals height, so a single sphere touches floor and oculus.

03 Constructing a model with the TI-Nspire

VIDEO **Modeling the Pantheon interior, keystroke by keystroke**
content.jwplatform.com/players/dfW6CJ9M-Y4Ue9e9A.html

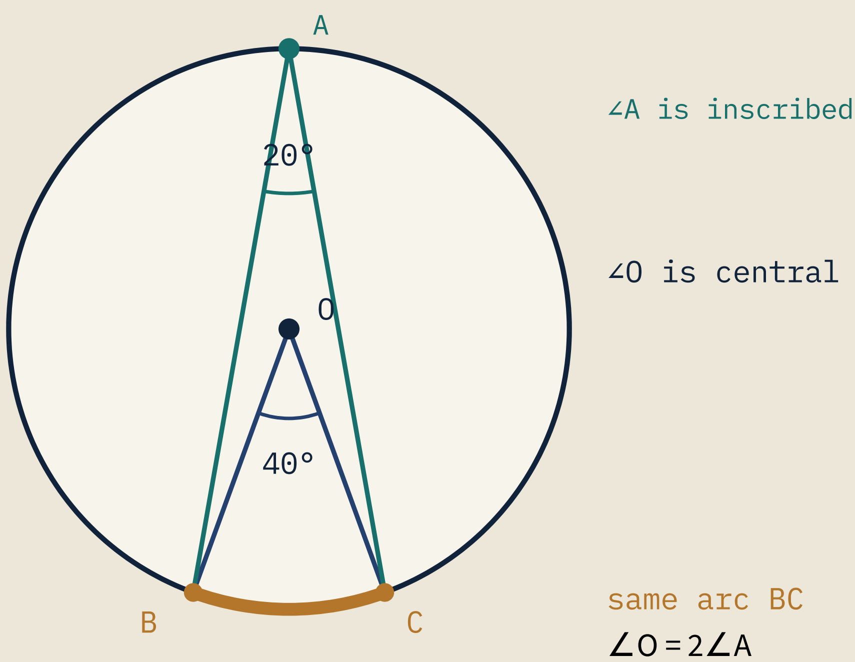
THE INSCRIBED ANGLE THEOREM — NAME IT

The video demonstrates the relationship but never states it. It has a name and a number, and it is one of the most useful facts in circle geometry:

$$\text{inscribed angle} = \tfrac{1}{2} \times \text{central angle}$$

when both open onto the *same* intercepted arc.

The proof sketch, if you want it: draw the radius to the inscribed angle's vertex. It splits the figure into isosceles triangles — two radii are always equal — and an exterior angle equals the sum of the two remote interior angles, which are equal to each other. Each piece of the central angle is double its piece of the inscribed angle.



Move A anywhere on the major arc and it still measures 20°.

TEST IT ON THE SKETCH

Have students drag A around the circle with B and C fixed. The inscribed angle does not budge. That invariance is the surprising part, and it is worth dwelling on: every seat on one side of a circular room sees the same arc of wall at the same angle.

04 The significance of the oculus

VIDEO **The Oculus and what it communicated**
content.jwplatform.com/players/XybELPSw-Y4Ue9e9A.html

ONE WINDOW, AND HOW SMALL IT IS

Oculus area = $\pi \times 4.1^2 \approx$ **52.8 m²**. Floor area = $\pi \times 21.65^2 \approx$ **1,473 m²**. The single opening is about **3.6%** of the floor — one part in twenty-eight. Everything visible inside the Pantheon for nineteen hundred years has been lit through a hole that small.

05 A better model: arcs and radians

VIDEO **A more accurate model of the Oculus**
content.jwplatform.com/players/DDehi1sb-Y4Ue9e9A.html

Radian	The angle whose intercepted arc is exactly one radius long. A full turn is 2π radians.
Degree ↔ radian	$180° = \pi$ radians. Multiply by $\pi/180$ one way, $180/\pi$ the other.
Arc length	$s = r\theta$, with θ *in radians*. This formula is the whole reason radians exist.

Worth saying explicitly: degrees are arbitrary — 360 is inherited from Babylon. Radians are defined so arc length is just radius times angle, with no conversion factor. Students who get this stop treating radians as an annoying second unit.

The oculus as a chord

$$\theta = 2 \cdot \arcsin(4.1/21.65) \approx 21.8° \approx 0.381 \text{ rad}$$

$$s = r\theta = 21.65 \times 0.381 \approx 8.25 \text{ m}$$

The arc is 8.25 m, the chord 8.2 m — 5 cm apart over 8 metres. For small angles an arc and its chord are nearly equal, which is why the curved rim of the oculus looks straight in a photograph.

> NOW THE COFFERS
>
> Five rings of 28. Twenty-eight equal divisions give $360/28 \approx 12.86° = 0.2244$ rad. At the springing line ($r = 21.65$ m) each coffer spans $s = 21.65 \times 0.2244 \approx$ **4.86 m**. Check the other way: circumference $\pi \times 43.3 \approx 136$ m, and $136/28 = 4.86$ m. Two routes, same answer — a good habit to model.

06 Coming full circle: where the light lands

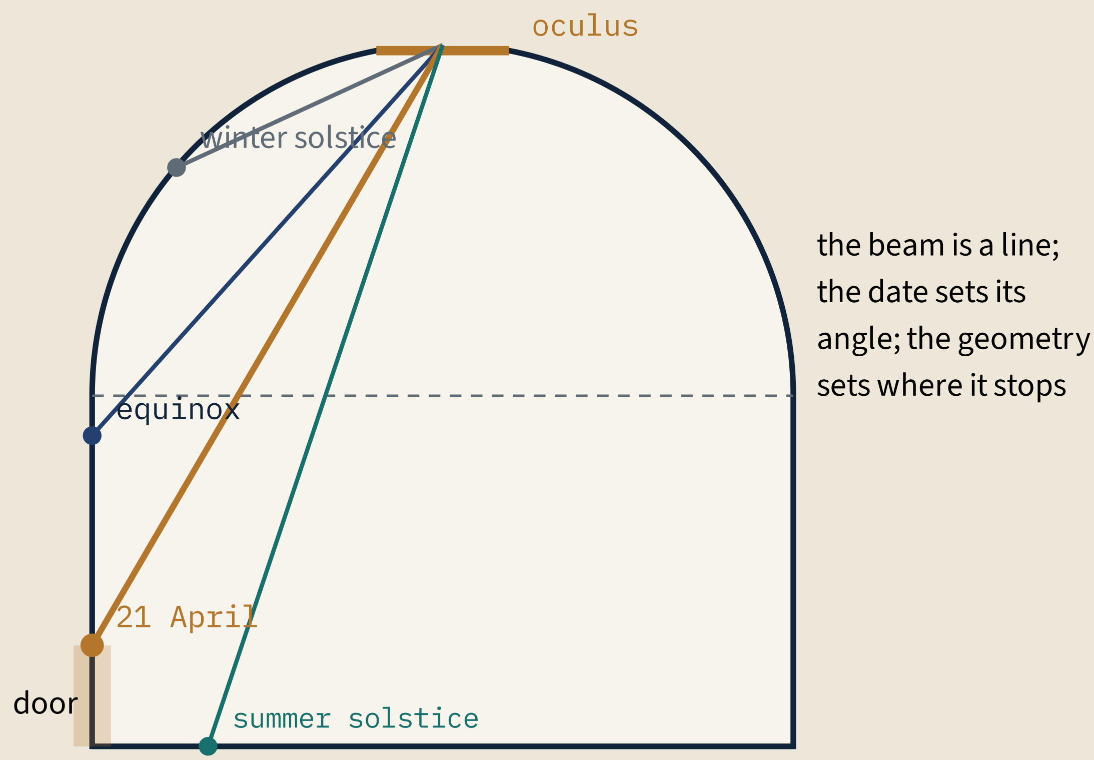
VIDEO **The intercepted arc and the key dates**
content.jwplatform.com/players/16shKypR-Y4Ue9e9A.html

Model the sunbeam as a straight line entering at the center of the oculus, 43.3 m up, and follow it until it hits something. One number comes from outside geometry — the noon sun altitude at Rome, latitude 41.9° N:

$$\text{altitude} = 90° - 41.9° + \delta$$

where δ is the sun's **declination**, running from −23.4° at the winter solstice to +23.4° at the summer solstice and passing 0° at each equinox. After that it is a right triangle: the beam drops $x \cdot \tan(\text{altitude})$ for every x metres sideways.

DATE	Δ	NOON ALTITUDE	BEAM LANDS
Winter solstice	−23.4°	24.7°	high on the dome, 35.8 m up
Equinoxes	0°	48.1°	the wall at 19.2 m — right at the springing line
21 April and 22 August	+11.6°	59.7°	the wall at 6.2 m — the doorway
Summer solstice	+23.4°	71.5°	the floor, 14.5 m from center



One opening, four dates, four very different places for the light to fall.

WHY EVERY SPOT HAS TWO DATES

Look at the highlighted row: 21 April *and* 22 August. Not a coincidence — it falls out of the geometry. Where the beam lands depends only on the altitude, which depends only on δ. But δ climbs from −23.4° in December to +23.4° in June and runs back down again, so every value except the two solstice extremes is reached **twice a year**, at dates roughly symmetric about 21 June.

The sun cannot tell you which of the two days it is; only a calendar can. That is a real limitation of every solar alignment ever built, and a building that works beautifully as a clock is a genuinely ambiguous calendar. Good discussion material with a history teacher.

07 Practice, with answers

1. A central angle measures 78°. What does an inscribed angle on the same arc measure? What if the inscribed angle is 31°?

ANSWER Inscribed = ½ × 78° = **39°**. Reversed, a 31° inscribed angle sits on a central angle of **62°**. The doubling works both directions provided both angles open onto the same arc.

2. Convert 12.86° to radians, then find the arc each coffer spans at the springing line.

ANSWER 12.86° × π/180 ≈ **0.2244 rad**; s = 21.65 × 0.2244 ≈ **4.86 m**.

3. A chord 12 m long is drawn in the cross-section circle. Find its central angle and minor arc.

ANSWER θ = 2 · arcsin(6/21.65) ≈ **32.2°** = 0.562 rad; arc = 21.65 × 0.562 ≈ **12.2 m** — just longer than the chord, as it must be.

4. On a day when the declination is −10°, where does the noon beam land?

> ANSWER Altitude = 90 − 41.9 − 10 = **38.1°**; tan ≈ 0.784. At the wall (21.65 m out) the beam has dropped 17.0 m, landing 43.3 − 17.0 = **26.3 m up** — above the springing line, so on the dome. Low winter sun puts light high, high summer sun puts it low. This inversion surprises almost everyone; let students predict before computing.

---

5. What declination puts the noon beam exactly at the center of the floor?

> ANSWER It would need altitude 90°, so δ = 90 − 48.1 = **41.9°** — Rome's latitude. But δ never exceeds 23.4°, so **it never happens**. The Pantheon sits too far north for the sun to stand overhead; the disc always lands off-center.

---

6. The oculus is 3.6% of the floor area. If you doubled its diameter, what fraction would it be?

> ANSWER **14.4%** — four times as much, not twice, since area scales with the square of the diameter. It would also remove far more concrete from the top of the dome, where the oculus does structural as well as optical work.

BACK TO THE OPENING QUESTION

Is there a day when the light lands on the front door? Yes, and students can find it without a historical source. Set the landing height to about 6 m up the north wall, solve backwards for the altitude (59.7°), subtract for the declination (+11.6°), and look up which days have it: around **21 April and 22 August**. The first is the date Romans celebrated as their city's founding.

The geometry cannot say whether Hadrian's architects planned it. What it can say is that if they wanted light on the door at noon, the height of the oculus and the width of the room left them exactly one choice of angle — and the calendar did the rest.

Teacher's Guide · Google Earth Voyager Story: Circular Structures, Part 2
Companion to the online lesson. Videos and the Voyager story are linked by URL above; the interactive version carries them inline.